Name: KEY **School:** ____________ **Team:** ____________

Simplify final answers and place them in the space given.

1. (2 points) Consider integers a and b such that $0 < a \leq b$, **what are the three smallest possible values of $a^3 + b^4$?**

Solution: We consider the smallest possible values of a and b, remembering that $0 < a \leq b$:

a	b	$a^3 + b^4$
1	1	2
1	2	17
2	2	24

Answer $2, 17, 24$

2. (3 points) **What is the number in the one's place of**

$$1^9 + 2^8 + 3^7 + 4^6 + 5^5 + 6^4 + 7^3 + 8^2 + 9^1?$$

Solution: For each term in the sum, we compute the individual one's place:

value	one's digit	value	one's digit
1^9	1	6^4	6
2^8	6	7^3	3
3^7	7	8^2	4
4^6	6	9^1	9
5^5	5		

The sum of these values is

$$1 + 6 + 7 + 6 + 5 + 6 + 3 + 4 + 9 = 47.$$

Which makes the one's digits of the larger number also 7.

Answer 7

3. (4 points) Use each of the digits $2, 3, 4, 6, 7, 8$ exactly once to construct two three-digit integers M and N so that $M - N$ is positive and as small as possible. **What is $M - N$?**

Solution: If $M-N$ is to be as small as possible we want to do our best to get the hundreds, tens, then ones digit close to one another. The best we can do with hundreds digit is within 1. So now we want the tens and ones digit of M to be as small as possible and the tens and ones digit of N to be as big as possible. This happens when $M = 426$ and $N = 387$.

Answer 39

University of Wisconsin-Eau Claire | Math Meet 2024
Event 2: Geometry | February 17, 2024

Name: KEY **School:** ____________ **Team:** ____________

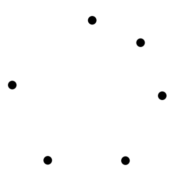
Simplify final answers and place them in the space given.

1. (2 points) The length of the arc on a circle intercepted by a 30° angle is 25 inches. **What is the circumference of the circle?** Your answer MUST include the units.

> **Solution:** This arc is $\frac{30}{360} = \frac{1}{12}$ of the circle. Thus the whole circle has length $25 \cdot 12 = 300$ inches.

Answer 300 **in=25 ft =8.3 yds**

2. (3 points) Consider six points arranged randomly in a circle. Draw a line segment between each pair of these points. **What is the maximum number of points of intersection that this collection of lines can have on the interior of the circle?**



> **Solution:** Any intersection exactly corresponds to selecting the four endpoints of the joined lines. Thus it is sufficient to select the four endpoints using $\binom{6}{4}$.

Answer $\binom{6}{4} = 15$

3. (4 points) Let A, B, C, and D be consecutive vertices of a regular polygon. If we know that $\angle ACD = 72°$, **how many sides does the polygon have?**

> **Solution:** If $x = \angle ABC = \angle BCD$ and $y = \angle BAC = \angle BCA$, then
>
> $$x - y = 72° \quad \text{and} \quad x + 2y = 180°.$$
>
> Using these two equations, $x = 108°$.
>
> Thus we are looking for the regular polygon all of who's interior angles are 108°. This would mean that
>
> $$180°(n-2) = 108°n.$$
>
> Solving for n results in $n = 5$.

Answer 5

University of Wisconsin-Eau Claire — Math Meet 2024

Event 3: Intermediate Algebra — February 17, 2024

Name: KEY **School:** ______ **Team:** ______

Simplify final answers and place them in the space given.

1. (2 points) Consider the line through points $(\beta, -9)$ and $(7, \beta)$ with slope β. **What is the value of β (list all, if any)?**

Solution: The slope of the line is

$$\beta = \frac{\beta - (-9)}{7 - \beta}.$$

Multiplying by the denominator yields $\beta(7-\beta) = \beta + 9$. Distributing the β and arranging to have 0 on one side give the quadratic equation

$$0 = \beta^2 - 6\beta + 9 = (\beta - 3)^2.$$

Therefore, the only solution is $\beta = 3$.

Answer 3

2. (3 points) Let α be a solution to the equation $x^4 + x^2 - 1 = 0$. **What is the value of $\alpha^6 + 2\alpha^4$?**

Solution: Since $\alpha^4 + \alpha^2 - 1 = 0$, we can write $\alpha^4 + \alpha^2 = 1$. Multiplying both sides of this equation by α^2 tells us that $\alpha^6 + \alpha^4 = \alpha^2$. Thus

$$\alpha^6 + 2\alpha^4 = \alpha^6 + \alpha^4 + \alpha^4 = \alpha^2 + \alpha^4 = 1$$

Answer 1

3. (4 points) **What are all of the possible solutions to the equation**

$$-\sqrt{2x-7} - 1 = -2\sqrt{x-4}?$$

Solution: To solve this equation, we can square both sides to obtain the equation

$$2x - 7 + 2\sqrt{2x-7} + 1 = 4(x-4).$$

Because there is still a square root in here, we need to rearrange and square again. The rearrangement will be obtained by subtracting the $2x-7+1$ from both sides then dividing both sides by 2:

$$\sqrt{2x-7} = x-5.$$

Squaring this results in

$$2x-7 = x^2-10x+25 \Leftrightarrow 0 = x^2-12x+32 = (x-4)(x-8).$$

Now we might assume that $x=4$ and $x=8$ are both answers. But in fact, the squaring operation can introduce new solutions that did not exist before. Thus we must check if either or both of these are solutions by inserting them into the original equation.

$$-\sqrt{2*8-7}-1 = -2\sqrt{8-4} \quad \text{and} \quad -\sqrt{2*4-7}-1 \neq -2\sqrt{4-4}$$

Answer 8

Name: KEY **School:** ________ **Team:** ________

Simplify final answers and place them in the space given.

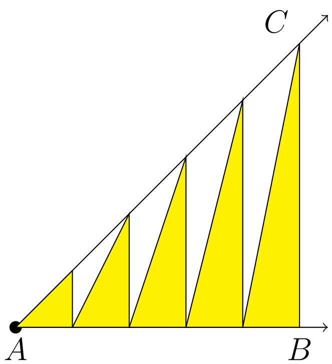
1. (2 points) **Compute** $\dfrac{\log(8)}{\log\left(\frac{1}{8}\right)}$**.**

Solution:

$$\frac{\log(8)}{\log\left(\frac{1}{8}\right)} = \frac{\log(8)}{\log(8^{-1})} = \frac{\log(8)}{-\log(8)} = -1$$

Answer -1

2. (3 points) In the diagram, the ray $\vec{AB}$ is horizontal and the ray $\vec{AC}$ has slope one. Each of the shaded triangles is a right triangle and has a base of length 1. **What percentage of the area of the triangle $\Delta(ABC)$ is covered up with these 5 shaded triangles?** Your answer MUST be a percent.



Solution:

The area filled by the shaded triangles is half of that for the region of boxes who's diagonals are the hypotenuses of the triangles. Thus the area of the shaded triangles is

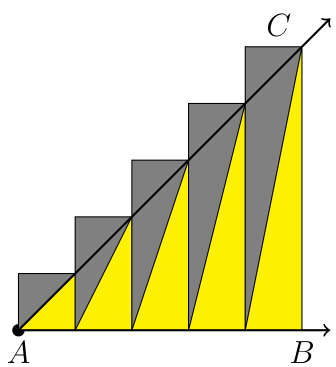
$$\frac{1}{2}(1\cdot 1 + 2\cdot 1 + 3\cdot 1 + 4\cdot 1 + 5\cdot 1) = \frac{15}{2}.$$

The area of $\Delta(ABC)$ is $\frac{1}{2}\cdot 5\cdot 5 = \frac{25}{2}$.

Therefore, the ratio of $\Delta(ABC)$ that is shaded is

$$\frac{15/2}{25/2} = \frac{15}{25} = \frac{3}{5}.$$

As $\frac{3}{5} = 0.6$, the percent shaded is 60%.



Answer 60%

3. (4 points) **What is the exact value of** $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{\cdots}}}}}$**?**

> **Solution:** Write $a = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{\cdots}}}}}$. Then
>
> $$a^2 = 2 + a.$$
>
> This can be rearranged so that $0 = a^2 - a - 2 = (a-2)(a+1)$. Thus $a = 2$ or $a = -1$. Since $a > 0$, we must have $a = 2$.

Answer 2

University of Wisconsin-Eau Claire — Math Meet 2024

Team Event — February 17, 2024

School: **KEY** **Team:** ____________

Simplify final answers and place them in the space given.

1. (10 points) Place the numbers 1, 2, 4, 8, 16, 32, 64, 128, 256 into a 3×3 grid such that

   - every number is placed exactly one
   - the product of numbers in any row or column is the same

   What is the value of the product in each row and column?

Solution: Since every number needs to be placed, the product of all of the rows is

$$1 \cdot 2 \cdot 4 \cdot 8 \cdot 16 \cdot 32 \cdot 64 \cdot 128 \cdot 256 = 2^{36}.$$

Let's call the answer A. Since each row must have the same product, namely A, we know that

$$A^3 = 2^{36}.$$

In particular, $A = 2^{12}$. Two possible solutions are

1	256	16
32	2	64
128	8	4

8	256	2
4	16	64
128	1	32

Answer $2^{12} = 4096$

2. (10 points) There are multiple values of n such that $n+70$ and $n-50$ are both perfect squares. **Find the sum of all of them.**

Solution: Let $n + 70 = a^2$ and $n - 50 = b^2$. Then $a^2 - b^2 = 120$, so $(a + b)(a - b) = 120$. Then $c = a + b$ and $d = a - b$ are both factors of 120. Consider the possible factorizations of 120:

c	d	
120	1	
60	2	$*$
40	3	
30	4	$*$
24	5	
20	6	$*$
15	8	
12	10	$*$

Notice that $c + d = (a + b) + (a - b) = 2a$. Therefore, c and d must be either both even or both odd. This rules out four of the possible factorizations, leaving only the possibilities listed with $*$.

This gives three possible solutions:

$a + b = 60$ and $a - b = 2 \Rightarrow 2a = 62 \Rightarrow a = 31 \Rightarrow n + 70 = 31^2 \Rightarrow n = 891$

$a + b = 30$ and $a - b = 4 \Rightarrow 2a = 34 \Rightarrow a = 17 \Rightarrow n + 70 = 17^2 \Rightarrow n = 219$

$a + b = 20$ and $a - b = 6 \Rightarrow 2a = 26 \Rightarrow a = 13 \Rightarrow n + 70 = 13^2 \Rightarrow n = 99$

$a + b = 12$ and $a - b = 10 \Rightarrow 2a = 22 \Rightarrow a = 11 \Rightarrow n + 70 = 11^2 \Rightarrow n = 51$

Therefore, the answer is $891 + 219 + 99 + 51 = 1260$.

Answer 1260

3. (10 points) **Find the exact value of $\sum_{k=0}^{2024} \sin\left(\frac{\pi}{3} + \frac{k\pi}{2}\right)$.** Your answer MUST be exact.

Solution: Since for all a, we have $\sin(a) + \sin(a + \pi) = 0$, a lot of terms in this sum cancel. In particular the sum of every four consecutive terms is 0:

$$\sin\left(\frac{\pi}{3}\right) + \sin\left(\frac{5\pi}{6}\right) + \sin\left(\frac{4\pi}{3}\right) + \sin\left(\frac{11\pi}{6}\right) = 0.$$

Therefore,

$$\sum_{k=0}^{2024} \sin\left(\frac{\pi}{3} + \frac{k\pi}{2}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}.$$

Answer $\sqrt{3}/2$

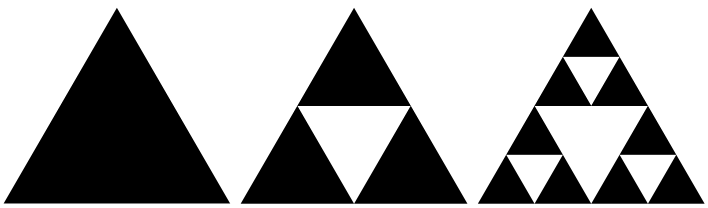
4. (10 points) Continuing the following process infinitely times will result in the geometric object called the Sierpinski Triangle.

Stage 0: Start with a solid equilateral triangle with area one.

Stage 1: Take the triangle from Stage Zero, divide it into four equilateral triangles and remove the middle triangle (see image below).

Stage 2-∞: Divide each solid triangle from the previous stage into four equilateral triangles and remove the middle triangle (see image below).

What is the area of all the solid triangles in Stage 4? Write your answer either as a fraction in reduced form or as an EXACT decimal representation.



Stage 0 Stage 1 Stage 2

Solution: At stage 0, the area is 1. At stage 1, the area is $\frac{3}{4}$, because a quarter of the area has been removed. Similarly, at stage 2, the area is $\frac{3}{4} \cdot \frac{3}{4} = \frac{9}{16}$, because what remains is $3/4$ of what was there. This process continues, the area at stage 3 is $\left(\frac{3}{4}\right)^3$, and at stage 4, the area is $\left(\frac{3}{4}\right)^4$.

Answer $\left(\frac{3}{4}\right)^4 = \frac{81}{256} = 0.31640625$

5. (10 points) A telegraph operator charges by the letter: 1¢ for each a, 2¢ for each b, 3¢ for each c, etc. Julia is telegraphing an important but sensitive six-letter message to Cesar. To prevent the telegraph operator from reading it, she shifts each letter thirteen places before sending it. (That is, a becomes n, b becomes o, …, m becomes z, n becomes a, o becomes b, and so on.) When she's done, she realizes it will cost exactly as much to send this encoded message as it would have cost to send the original message. **How many different messages could Julia be sending?**

Solution: The cost of the letters a through m increases by 13¢, and the cost of the letters n through z decreases by 13¢. So, the message must have three letters from the former set and three letters from the latter set. Allowing repeats, there are 13^3 trios from a through

m and 13^3 trios from n through z. Then there are 6 choose 3 ways of placing the latter trio among the six spots. That gives us

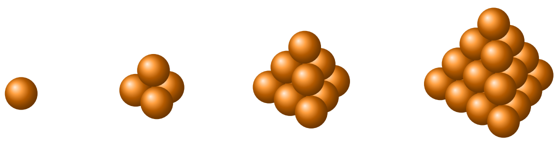
$$13^3 \cdot 13^3 \cdot \frac{6!}{3!(6-3)!} = 20 \cdot 13^6$$

possible messages.

Answer $20 \cdot 13^6 = 96\,536\,180$

6. (10 points) Consider the sequence composed of the number of oranges needed to stack a perfect pyramid.

How many rows of oranges are needed to make a pyramid of 2024 oranges?



1 4 10 20

Solution: Each new row adds a triangle with $1+2+\cdots+k = \frac{k(k+1)}{2} = \binom{k+1}{2}$ oranges. So now we're asking for what value of n is

$$\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{n+1}{2} = \binom{n+2}{3} = 2024.$$

This happens when $n = 22$.

Alternately, we see that if x is the number of rows of oranges, then the number of oranges is

$$f(x) = 1/6x^3 + 1/2x^2 + 1/3x,$$

and $f(22) = 2024$.

Fun Math: These are called *tetrahedral numbers*. And the step from one to the next is given by the *triangular numbers*.

Answer 22