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Message from Provost and Vice Chancellor Patricia A. Kleine

Since 1988, UW-Eau Claire has been recognized as the University of Wisconsin System's *Center of Excellence for Faculty and Undergraduate Student Research Collaboration*. In 2016, UW-Eau Claire received the National Council on Undergraduate Research's (NCUR's) campus award for *Undergraduate Research Accomplishments*. In 2023, UW-Eau Claire will host NCUR's annual conference which will bring to campus 3,000-4,000 faculty and student researchers.

No program exemplifies the institution's commitment to opportunities in undergraduate research for students more than UW-Eau Claire's Ronald E. McNair Post-Baccalaureate Achievement Program. Because of the high academic quality of students' studies, McNair scholars have been named Fulbright, Goldwater, Truman, and Rhodes Scholars.

This journal presents the culmination of two years of students working with their faculty mentors on critical questions in their disciplines and preparing their research for professional publication and presentation.

To the students, congratulations on the completion of your research projects, and best wishes for continued success in graduate school.

To my faculty colleagues, thank you for mentoring these remarkable students so well. I continue to celebrate your exceptional commitment to student success.

To the reader, I hope you enjoy reviewing this journal and the wealth and breadth of research within it.

Patricia A. Kleine
Provost and Vice Chancellor for Academic Affairs
Academic Year 2019-2020

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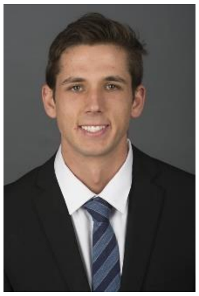

Hurt Feelings: The Role of Receivers' Personality in Emotional Reactions to Others' Words

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Abstract

The current study aims to determine how much consensus there is among perceivers about the perceived harmfulness of different types of statements. In the spring of 2019, we asked students on the UWEC campus to write down what other individuals have asked them or said to them that they thought were either intentionally or unintentionally hurtful. In addition to those intentionally hurtful and unintentionally hurtful statements, we generated a (baseline) set of clearly positive statements. In the fall of 2019, we randomly assigned 452 participants to one of three rating tasks. Participants assigned to Version 1 rated how hurt each statement made them feel (0=Not hurt at all, 100=Extremely hurt). Participants who get Version 2 rated how anxious each statement made them feel (0=not anxious at all, 100=Extremely anxious). Participants who get Version 3 rated the intent of the person's statement (0=They are NOT trying to be hurtful, 100=They are definitely trying to be hurtful). After rating the statements, participants completed a Words Can Harm questionnaire (Bellet et al., 2018) in which they rated their level of agreement with 10 statements focused on the idea that words, in general, can cause emotional harm. Results from the study showed strong consensus among participants in their reactions to the positive statements but far less consensus for the intentionally and unintentionally hurtful statements. Personality was important: participants who scored high in alienation and stress reaction perceived more hurt, anxiety, and intent to harm in the statements they read

Keywords: microaggressions, conversation, hurt, anxiety, intent.

Introduction

Microaggressions were first introduced to the broader discipline of psychology in the context of race relations and defined as "brief and commonplace daily verbal, behavioral, or environmental indignities, whether intentional or unintentional, that communicate hostile, derogatory, or negative racial slights and insults toward people of color" (Sue et al., 2007, p. 271). Over the past decade, awareness of microaggressions has increased and use of the term "microaggression" has spread beyond race into many domains, including gender and sexuality (Lilienfeld, 2017). In some universities, administrators and faculty distribute lists of words and phrases that students and staff are asked to refrain from using out of concern for their presumed harmful effects (Lukianoff & Haidt, 2018). Despite the good intentions of individuals on the frontlines of the microaggression movement, the research program on microaggressions has not provided (a) clear operational definitions of the microaggression construct; (b) rigorous evidence for the claim that microaggressions cause psychological harm to those who perceive themselves as micro-aggressed against; or (c) evidence that individuals agree about what types of statements are – and are not - harmful (Lilienfeld, 2017). Concern in American society about the potential harm of subtle slights and insults introduces the need for systematic data on *who* feels hurt by *what* types of statements. Thus, labeling too many things as a "microaggression" could backfire by essentially leading people not only to interpret ambiguous statements as harmful but also to self-censor (Lukianoff & Haidt, 2018).

Our lab (alongside other labs around the country; see Bellet, Jones, & McNally, 2018) is beginning a series of studies to address Lilienfeld's concerns about the microaggressions construct. In our first study, the aim was to illustrate that a clear operationalization of the term is necessary by showing, experimentally, that priming individuals to perceive others' words as harmful will lead them to perceive others' words as harmful as well as to report a lower likelihood of using those words themselves. Through a survey, participants were asked to rate several ambiguous and benign statements and questions (e.g., "What do you like to watch on Netflix?"). Half of the participants rated the statements on harmfulness (1= Harmless, 7=Harmful), and the other half rated the statements on their likelihood of using each of the statement (1=I am unlikely, 7=I am likely). Before rating the statements, however, participants were randomly given one of three versions of a set of introductory sentences. In the *control* condition, participants received an embedded phrase that read "in our everyday interactions with various people, we say and ask all kinds of things." In the *unintentional harm*, the phrase read "in our everyday interactions with various people, we sometimes say and ask things that are not meant to be harmful but that actually can be harmful and can create a hostile environment." In the *intentional harm* condition, participants read "in our everyday interactions with various people, we sometimes say and ask things that are intended to be harmful and create a hostile environment." The idea was to prime participants about harm before they read each statement and reported how harmful they perceived it to be. At the end of the questionnaire, participants completed a personality inventory to assess whether negative emotionality is related to individual differences in the degree to which people perceive various statements as harmful. Results from the study demonstrated the effects of priming participants, as the participants who were primed with the idea that people say unintentionally harmful words perceived the ambiguous statements as more harmful than those who were not primed at all. (However, those who were primed with the idea that people say intentionally harmful words did <u>not</u> differ from those who were not primed at all.) Notably, we also found that in the "unintentionally harmful" prime condition, participants who were higher in negative emotionality perceived ambiguous statements as more harmful. These results are consistent with Lilienfeld's concern that telling those who already view themselves and others in a negative to look out for more hurtful things people say might exacerbate unnecessarily such a trend to perceive the negative in others' words.

To extend this work, then, the objective of the current research is to determine if there is any consensus about what other people perceive as harmful when it comes to the things that people say to one another. We also want to determine whether people's perceptions of harm in statements – including those nominated by others as statements that are not meant to be harmful -- are related to their level of negative emotionality.

Method

Participants

Participants were recruited through the University of Wisconsin-Eau Claire Research Participation System (SONA), which is run by the Psychology Department but open to students across campus. Standard implied consent processes were followed. A total of 571 students participated. Of that initial sample, 115 participants were excluded for spending less than five minutes in the survey, a time we deemed necessary to give if participants were reading the materials and questions. An additional four participants were also excluded from the study because they completed less than 40% of the items. The final sample of participants totaled 452 participants: 150 participants provided *Anxious* ratings, 150 provided *Hurt* ratings, and 152 participants provided *Likelihood of Intention to Hurt* ratings.

Materials

Survey preparation. The research team first collected examples of statements that might hurt. During the spring of 2019, members of the team approached students on campus at the University of Wisconsin Eau-Claire and asked them to list either (a) things others had said to the them or that they had said to others that were *intentionally* harmful, or (b) things others had said to the them or that they had said to others that were *unintentionally* harmful. Three researchers then independently categorized the statements into content domains. The three researchers collectively agreed on 16 content domains, as listed in Table 1. For the purpose of shortening the survey and for subsequent data collection, one statement that had been nominated as intentionally hurtful and one statement that had been nominated as unintentionally hurtful were chosen from each domain. In addition to the intentional and unintentional statements, researchers generated statements that would be perceived as positive. In total, the survey consisted of 48 statements, three statements from each of 16 domains, one positive statement, one unintentionally hurtful statement, and one intentionally hurtful statement.

Three Surveys. The research team then constructed a questionnaire that was comprised of the 48 statements (16 domains with three types of statements for each domain), a brief personality inventory, and a brief attitude scale.

Participants were randomly assigned to one version of the statements portion of the questionnaire. For the first version, participants reported how hurt they would feel if someone said each of the statements to them (0= Not hurt at all hurtful, 100= Extremely hurt). Internal consistency was strong for all three types of statements (Intentional α= .92, Unintentional α= .89, and Positive α= .94). In other words, a participant who felt hurt in response to one intentionally hurtful statement was highly likely to also feel hurt in response to other intentionally hurtful statements. If a participant felt hurt in response to one unintentionally hurtful statement, they were highly likely to also feel hurt in response to other unintentionally hurtful statements. And, if a participant felt hurt in response to one of the positive statements, they were highly like to feel hurt in response to the other positive statements, as well.

In the second version, participants rated how anxious they would feel if someone if someone said each of the statements to them (0= Not anxious at all anxious, 100= Extremely Anxious). The internal consistencies were strong for all three types of statements

Table 1

Domain	Intentional	Unintentional	Positive
Personality	All you care about is yourself.	You think too much about what other people think of you.	You are very selfless.
General Ineptitude	You will never be able to do that.	You finally did it.	Great job.
Intelligence	Why are you so dumb?	You don't know that?	Only you could have answered a hard question like that.
Health/Sickness	You look ill.	You should see a doctor.	You are so lively.
Body	You are fat.	You should exercise more.	You are in really good shape.
Clothes/Fashion	I would never wear that.	That is different from what you usually wear.	I like the style of clothes you wear.
Face	Your teeth are crooked.	You should try Invisalign.	I love your smile.
Hair	Your hair looks awful.	Your hair is something else today.	Your hair looks so healthy.
Attractiveness	You're ugly.	You're getting to be so pretty.	You are good-looking.
Habits/Behaviors	You have no life.	You are not living the college experience.	You are so outgoing.
Hygiene	You smell.	You should shower.	You smell so good.
Dating failure	Why would anyone ever date you?	You have to be desirable to get a date.	Anyone would want to date you.
Social Belonging	No one wants to be your friend.	(Making friends might be a struggle for you.)	You're a great friend.
Social Choices	Your friends are lame.	I don't know why you are friends with them.	You have such great friends.
Minimizing	Stop making a big deal out of it.	It was just a joke.	Your feelings are definitely valid.
Personal Insults	Couldn't you just be normal?	(Can you just be cool this one time?)	You are so cool.

16 content domains with three types of statements for each domain: Intentionally hurtful, unintentionally hurtful, and positive.

(Intentional α= .93, Unintentional α= .90, and Positive α= .92). In other words, a participant who felt anxious in response to one intentionally hurtful statement was highly likely to also feel anxious in response to other intentionally hurtful statements. If a participant felt anxious in response to one unintentionally hurtful statement, they were highly likely to also feel anxious in response to other unintentionally hurtful statements. And, if a participant felt anxious in response to one of the positive statements, they were highly like to feel anxious in response to the other positive statements, as well.

Participants who were assigned to the third version rated how likely it was that the person who said the statements was trying to be hurtful (0= They are NOT trying to be hurtful, 100= They are DEFINITELY trying to be hurtful). Internal consistencies were again high for all three types of statements (Intentional α= .85, Unintentional α= .86, and Positive α= .80). In other words, a participant who felt that the person saying the statements was trying to be hurtful in response to one intentionally hurtful statement was highly likely to also feel that the person saying the statements was trying to be hurtful in response to other intentionally hurtful statements. If a participant felt that the person saying the statements was trying to be hurtful in response to one unintentionally hurtful statement, they were highly likely to also feel that the person saying the statements was trying to be hurtful in response to other unintentionally hurtful statements. And, if a participant felt that the person saying the statements was trying to be hurtful in response to one of the positive statements, they were highly like to feel that the person saying the statements was trying to be hurtful anxious in response to the other positive statements.

Words Can Harm Scale. All participants, regardless of dependent variable being measured, completed a 10-item Words Can Harm Scale (Bellet e al., 2018) that assessed the degree to which participants perceived harm in things that they read and how harmful what they read could be to others. Participants rated their agreement on a 100-point scale (0= strongly disagree, 100= strongly agree). The Words Can Harm scale had strong internal reliability in each subsample (Hurt sample α= .89, Anxious sample α= .88, and Intent ample α= .86).

Negative Emotionality Scale. All Participants also completed a 15-item questionnaire to assess their negative emotionality. The negative emotionality scale used had tree facets: aggression, alienation, and stress reaction. The anger facet was not included in the final analysis due to weak internal consistency (Hurt, α=.46, Anxious, α=.40, and Intent, α=.50). The other two facets were both included. Internal consistency was strong for stress reaction across conditions (Hurt, α=.75, Anxious, α=.62, and Intent, α=.72). Internal consistency was also strong for alienation (Hurt, α=.75, Anxious, α=.66, and Intent, α=.76).

Demographics. As a final section of the questionnaire, all participants reported their age, gender, area of study, and political orientation.

Results

Version 1. Over 90% of participants rated the *positive* statements (M= 2.45, SD= 7.70) as not at all hurtful, suggesting strong consensus that these statements are perceived as not hurtful at all (see Figure 1.1). Overall, participants rated the *intentionally* hurtful statements (M= 57.90, SD= 20.39) as moderately hurtful, but the histogram in Figure 1.2 shows substantial variability in responses across participants. That is, there was not strong consensus among participants in how hurt they felt by the statements. Overall, participants rated the *unintentionally* hurtful statements (M= 39.30, SD= 17.39) as somewhat hurtful. Figure 1.3 shows substantial variability in responses across participants. That is, there was not strong consensus among participants in how hurt they felt by the statements.

Table 2 shows correlations between the participants personality traits and how **hurt** they reported feeling in response to the statements. A statistically significant correlation was found between participants hurt ratings of the intentionally hurtful statements and their stress reaction scores (p < .001). Individuals who scored higher in stress reaction (anxiety) tended to report feeling *more* hurt by the intentionally hurtful statements. A statistically significant correlation was found between participants' hurt ratings of the unintentionally hurtful statements and their stress reaction scores (p < .001). Individuals who scored higher in stress reaction (anxiety) tended to report feeling *more* hurt by the unintentionally hurtful statements. A statistically significant correlation was also found between participants' hurt ratings of the intentionally hurtful statements and their words can harm score (p < .001). Individuals who scored higher in their belief that words can harm also tended to report feeling more hurt by the intentionally hurtful statements. A statistically significant correlation was found between participants' hurt ratings of the unintentionally hurtful statements and their words can harm score (p= .017). Alienation (feeling ostracized and victimized by others) was related to feeling hurt by positive statements (p = .017). There were no statistically significant correlations between participants' perceptions of hurt in response to the positive statements and their stress reaction scores, or their words can harm scores. There was also no statistically significant correlation between participants' perceptions of hurt in response to either the intentional or unintentional statements and their alienation scores.

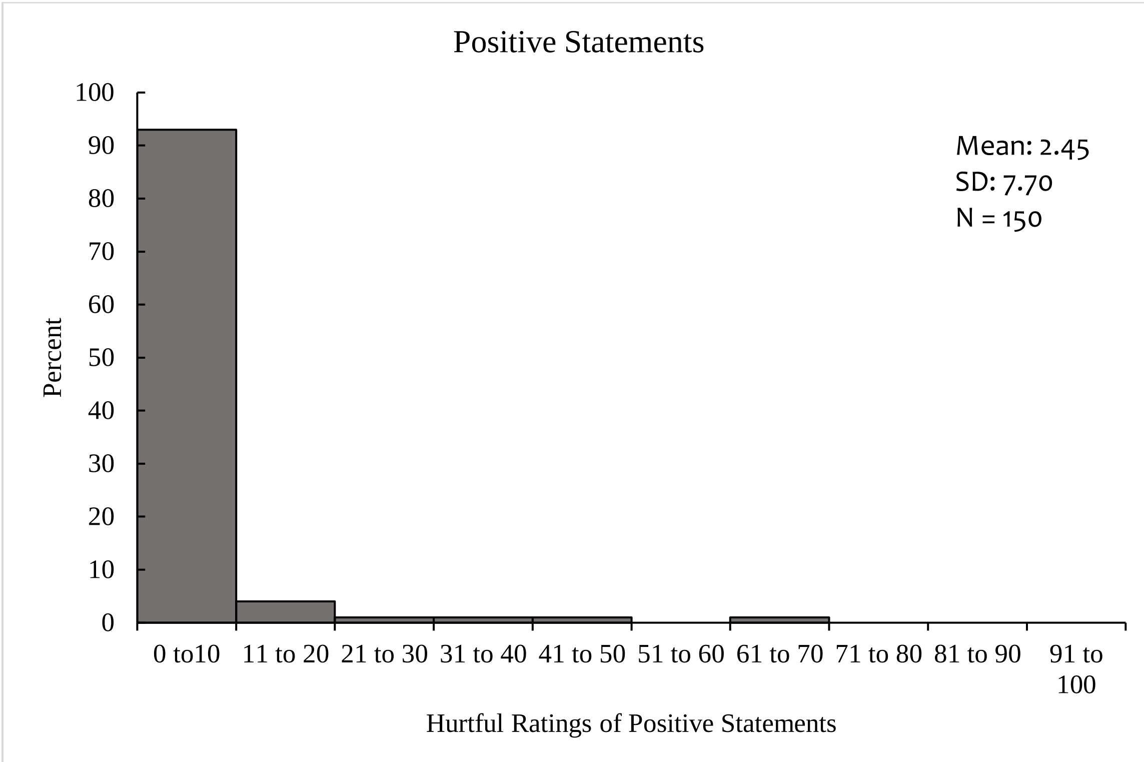

Figure 1.1: Hurt ratings in response to the positive statements

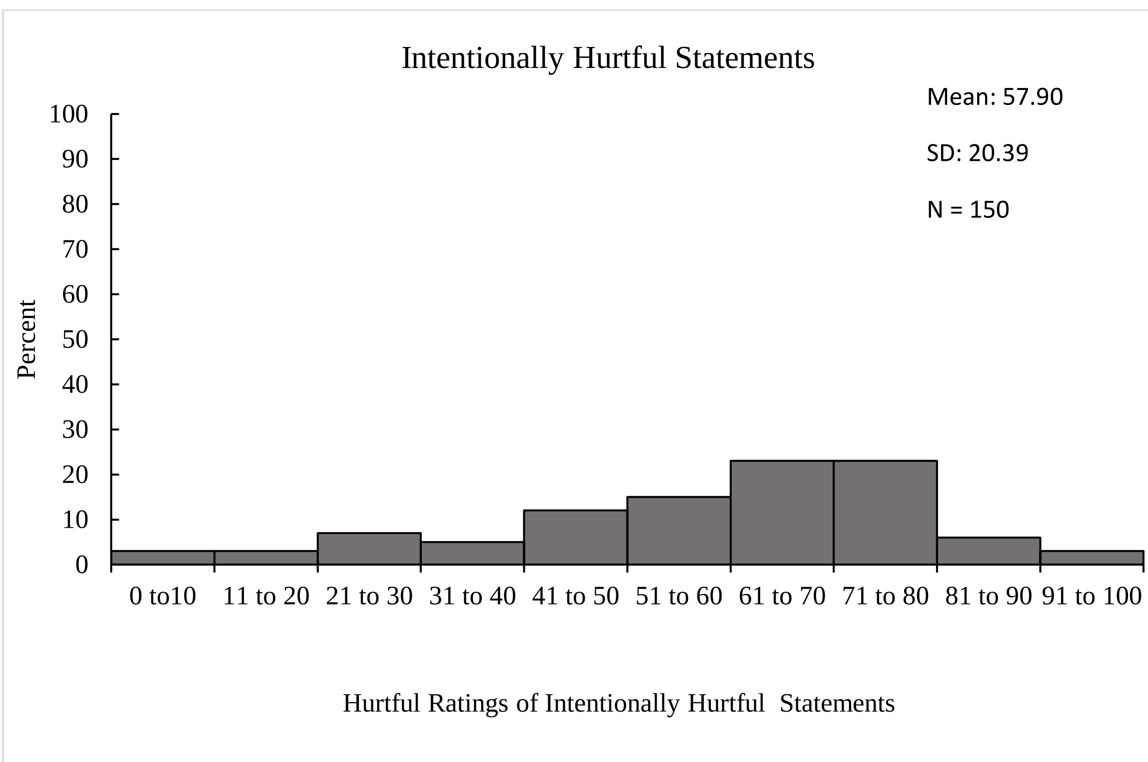

Figure 1.2: Hurt ratings in response to the intentionally hurtful statements.

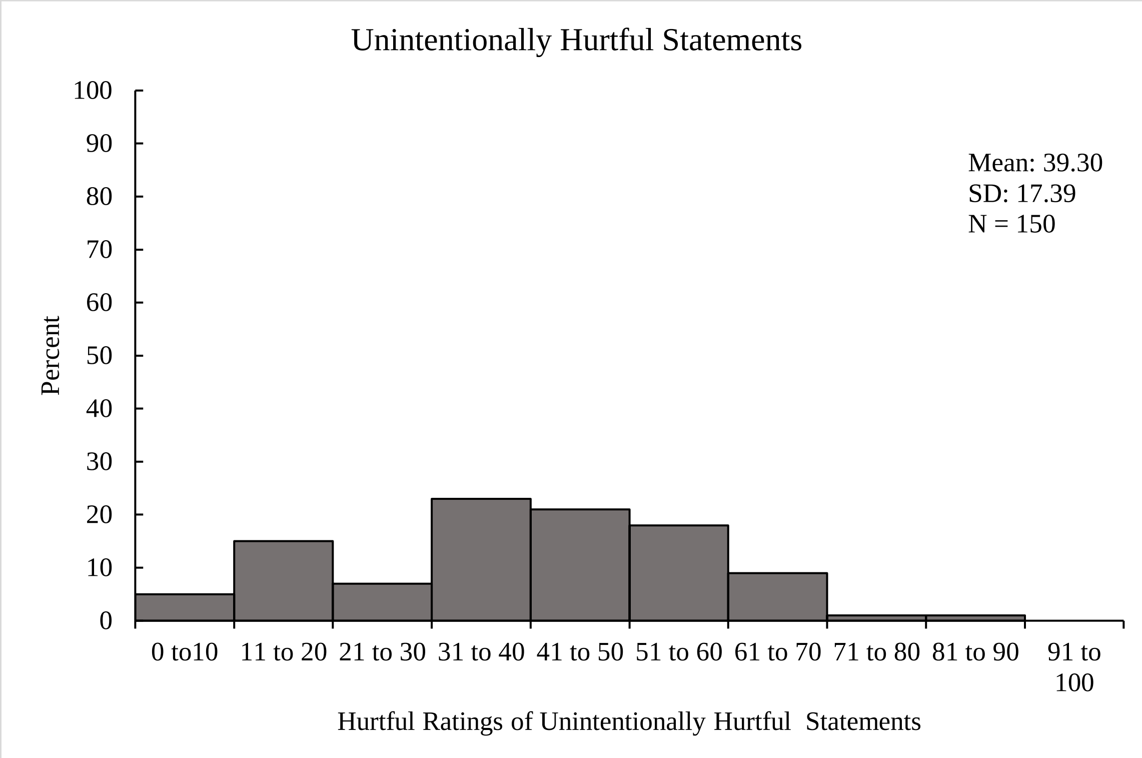

Figure 1.3: Hurt ratings in response to the unintentionally hurtful statements.

Table 2

	Positive Statements	**Intentionally Hurtful Statements**	**Unintentionally Hurtful Statements**
Stress Reaction	*r*= -.01 95% CI [.14, .43]	*r*= .33*** 95% CI [.18, .47]	*r*= .29*** 95% CI [.14, .43]
Alienation	*r*= .19* 95% CI [.04, .34]	*r*= .05 95% CI [-.11, .21]	*r*= .14 95% CI [-.02, .29]
Words can Harm	*r*= -.13 95% CI [-.30, .05]	*r*= .33*** 95% CI [.18, .47]	*r*= .29*** 95% CI [.14, .43]

Note. * $p < .05$, ** $p < .01$, *** $p < .001$. *Correlations between participants perceptions of hurt felt on each of the three different statements and their response to the negative emotionality facets and the words can harm scale.*

Version 2. Over 2/3 of participants reported that positive statements (*M*= 9.82, *SD*= 12.58) did not make them feel anxious at all. There was substantial consensus across participants that positive statements do not cause anxiety (see Figure 2.1). Overall, participants felt moderately-to-very anxious by the intentionally hurtful statements (*M*= 58.89, *SD*= 20.07). The histogram in Figure 2.2 shows substantial variability in responses across participants. That is, there was not strong consensus among participants in how anxious they felt by the statements. Overall, participants felt moderately anxious by the unintentionally hurtful statements (*M*= 48.40, *SD*= 18.21). The histogram in Figure 2.3 shows substantial variability in responses across participants. That is, there was not strong consensus among participants in how anxious they felt by the statements.

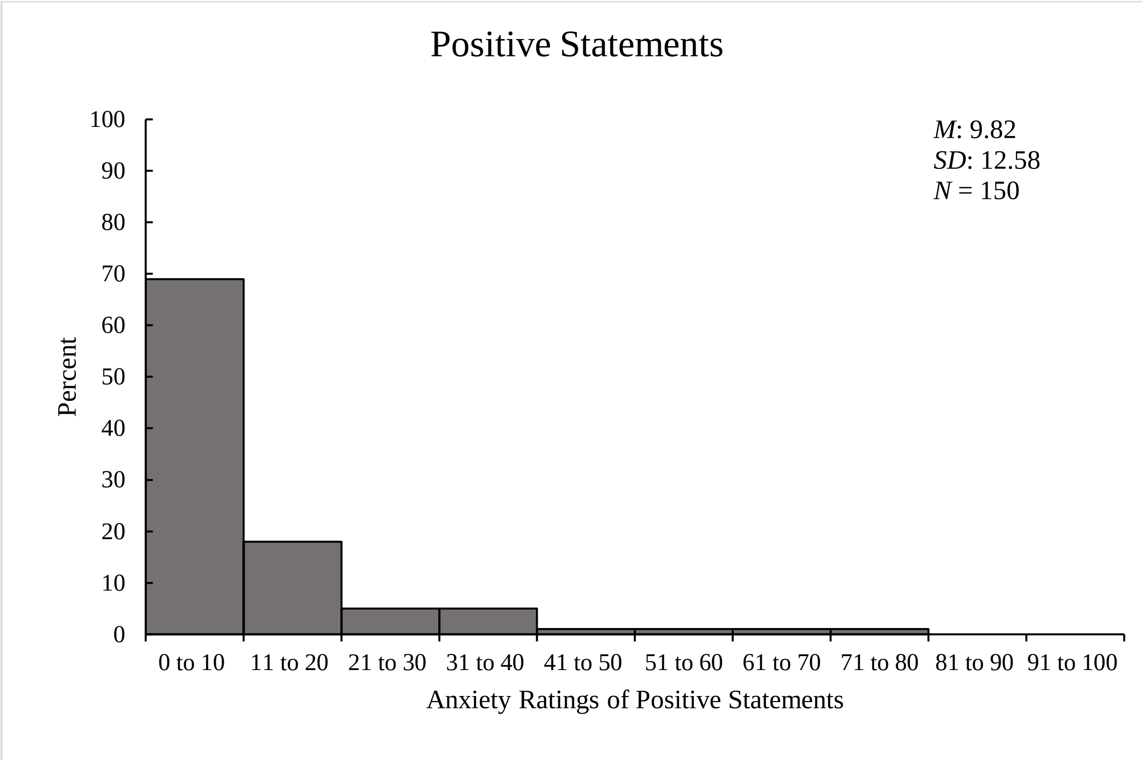

Figure 2.1: Anxious ratings in response to the positive statemetns.

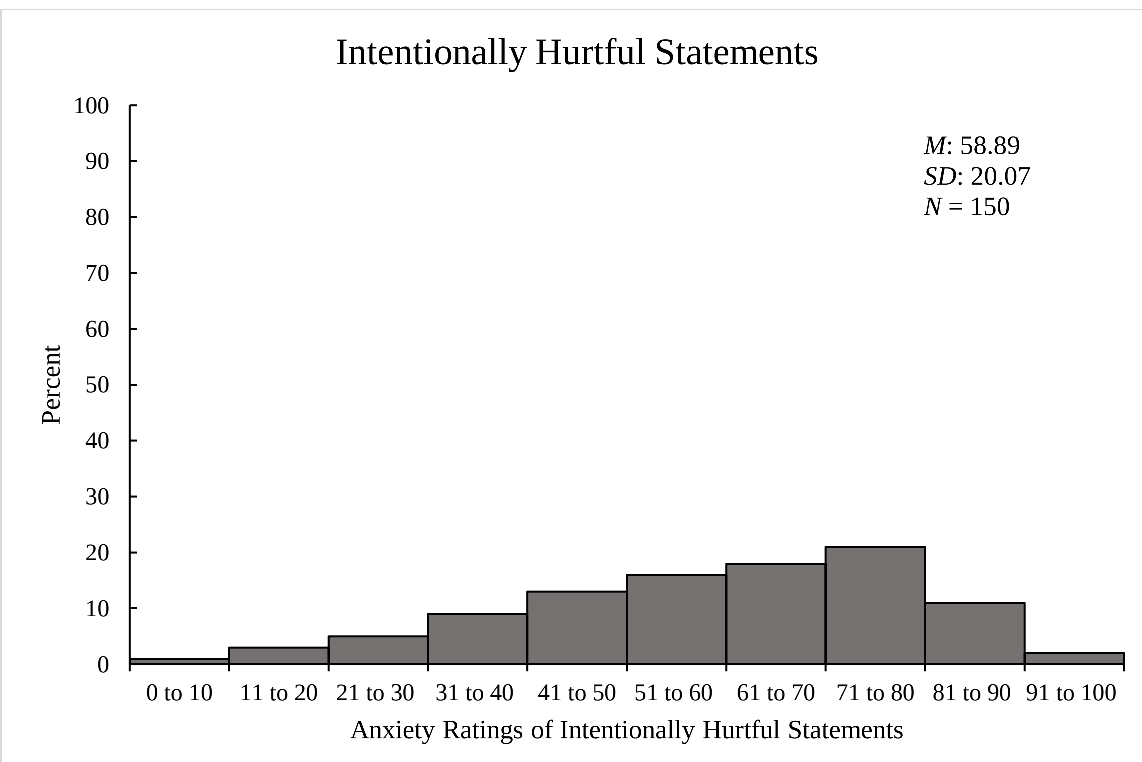

Figure 2.2: Anxious rating in response to the intentionally hurtful statements.

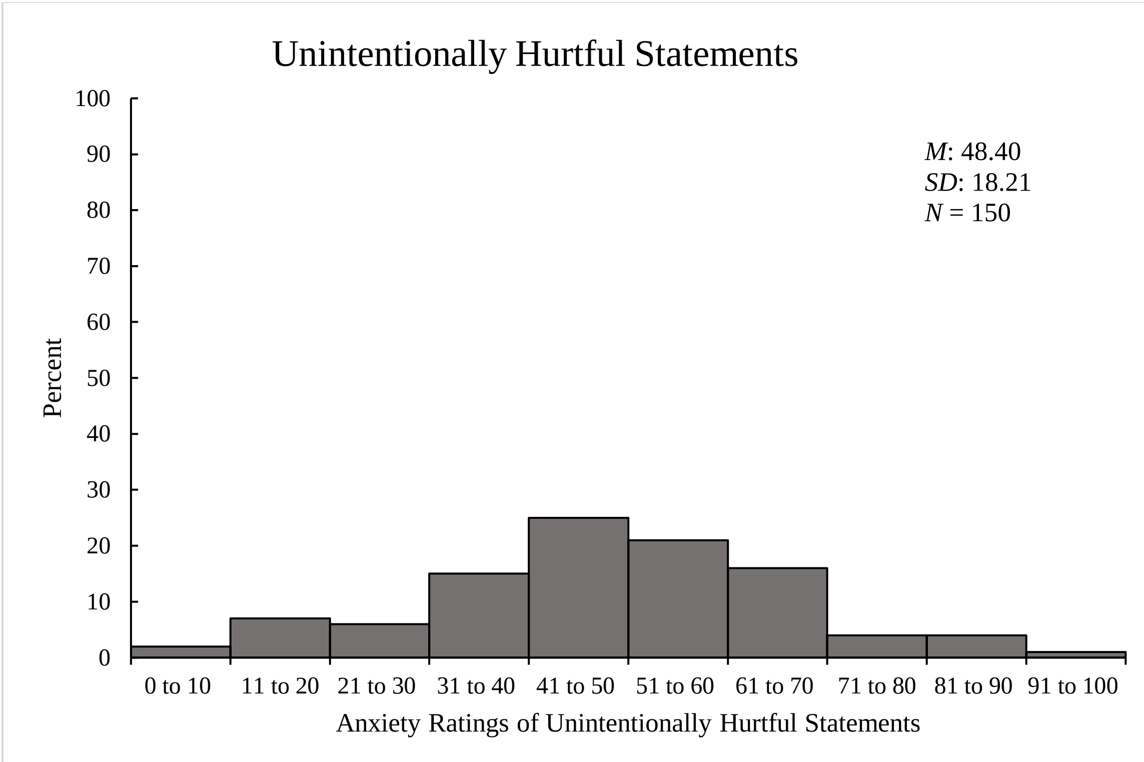

Figure 2.3: Participants rating on how anxious the unintentionally hurtful statements made them feel varied a lot from one participant to another with most participants ratings of the statements falling around the indifferent point.

Table 3

	Positive Statements	**Intentionally Hurtful Statements**	**Unintentionally Hurtful Statements**
Stress Reaction	r=.10 95% CI [-.06, .26]	r=.31*** 95% CI [.16, .45]	r= .34*** 95% CI [.19, .48]
Alienation	r=.37*** 95% CI [.22, .50]	r= .11 95% CI [-.05, .27]	r= .23* 95% CI [.07, .38]
Words can Harm	r= -.03 95% CI [-.02, .15]	r= .23* 95% CI [.06, .39]	r= .23* 95% CI [.06, .39]

Note. * $p < .05$, ** $p < .01$, *** $p < .001$. *Correlations between participant's perceptions of anxiety felt with each of the three different statements and their response to the negative emotionality facets and the words can harm scale.*

Table 3 shows correlations between participants personality traits and how anxious they reported feeling in response to the statements. A statistically significant correlation was found between participants' anxious ratings of the intentionally hurtful statements and their stress reaction scores ($p < .001$). Individuals who scored higher in stress reaction (anxiety) tended to report feeling *more* anxious by the intentionally hurtful statements. unintentionally hurtful statements. A statistically significant correlation was found between participants' anxious ratings of the unintentionally hurtful statements and their stress reaction scores, ($p < .001$). Individuals who scored higher in belief that words can harm tended to report feeling more anxious by the unintentionally harmful statements. A statistically significant correlation was also found between participants' anxious ratings of the positive statements and their alienation scores ($p < .001$). Individuals who scored higher in alienation also tended to report feeling more anxious in response to the positive statements. A statistically significant correlation was also found between participants' ratings of the unintentionally harmful statements and their alienation scores ($p = .006$). That is, individuals who scored higher in alienation also tended to report feeling more anxious in response to the unintentionally harmful statements. There were no statistically significant correlations between participants felt anxiety in response to the positive statements and their scores of either stress reaction or words can harm. No statistically significant correlation was found between participant's perception of anxiety felt in response to the intentionally harmful statements and the alienation score.

Version 3. In response to the positive statements (M= 4.57, SD= 6.36), nearly 90% felt that they were not at all likely to be delivered with intent to harm, suggesting a general consensus across participants that positive statements do not betray an intent to harm (see Figure 3.1). Overall, participants rated the intentionally hurtful statements (M= 75.65, SD= 12.58) as very likely to be delivered with the intent to be hurtful. The histogram in Figure 3.2 shows that there was some consensus among participants in the intent they perceived in the statements. Overall, participants rated the unintentionally hurtful statements (M= 48.47, SD= 14.63) as moderately likely to be delivered with the intent to be hurtful. The histogram in Figure 3.3 shows some consensus (but less than above) among participants in the intent they perceived in the statements.

Table 4 shows correlations between participants' personality traits and their rating of how likely the deliverer of the statement was intending to be hurtful**.** A statistically significant correlation was found between participants' perception of the likelihood that the person saying the statement was trying to be hurtful in response to the intentionally hurtful statements and their words can harm scale ($p < .001$). Individuals who scored higher in belief that words can harm tended to perceive greater likelihood of hurtful intent in response to the intentionally hurtful statements. A statistically significant correlation was also found between participants' perception of how likelihood that the person saying the statement was trying to be hurtful in response to the unintentionally hurtful statements and their words can harm scale ($p < .001$). Individuals who scored higher in belief that words can harm tended to perceive greater likelihood of hurtful intent in response to the unintentionally hurtful statements. A statistically significant correlation was found between participants alienation scores and their rating of perceiving intent to hurt from the person saying the statement and their response to the positive statements ($p = .013$). No statistically significant correlations were found between participants stress reaction scores and the ratings of the likelihood that the person saying the statement is trying to be hurtful in response to either the intentionally or unintentionally hurtful statements.

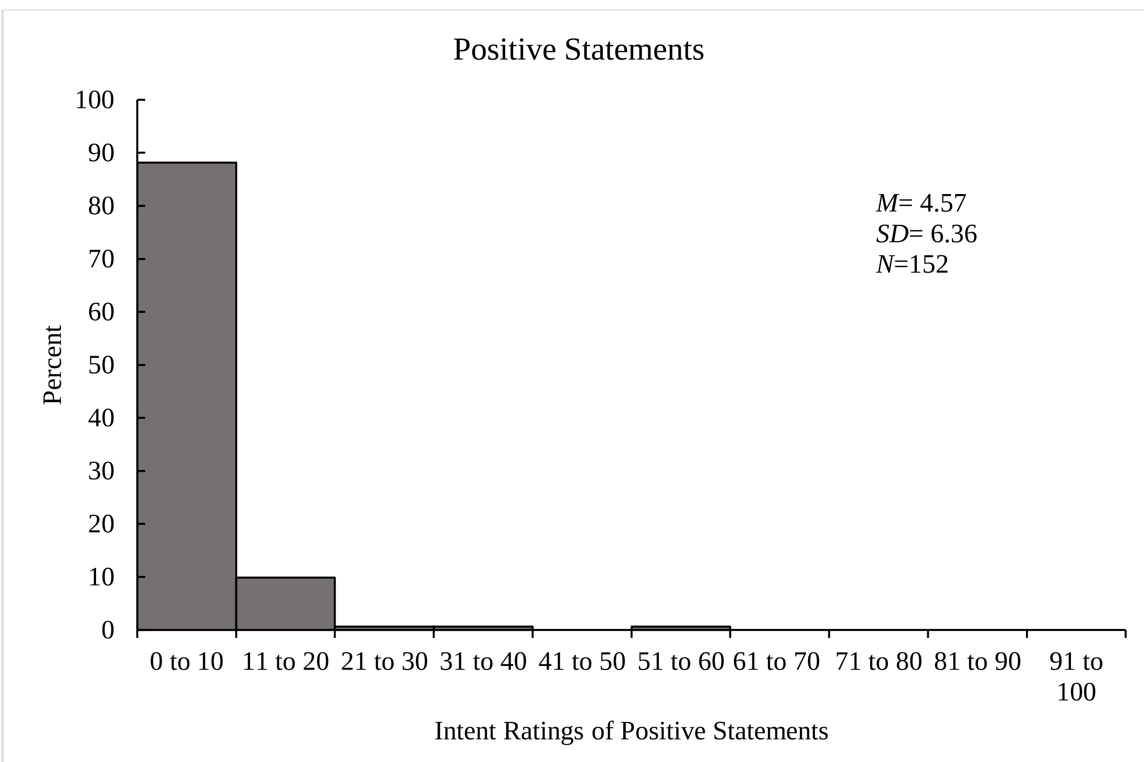

Figure 3.1: Intent ratings in response to positive statemetns.

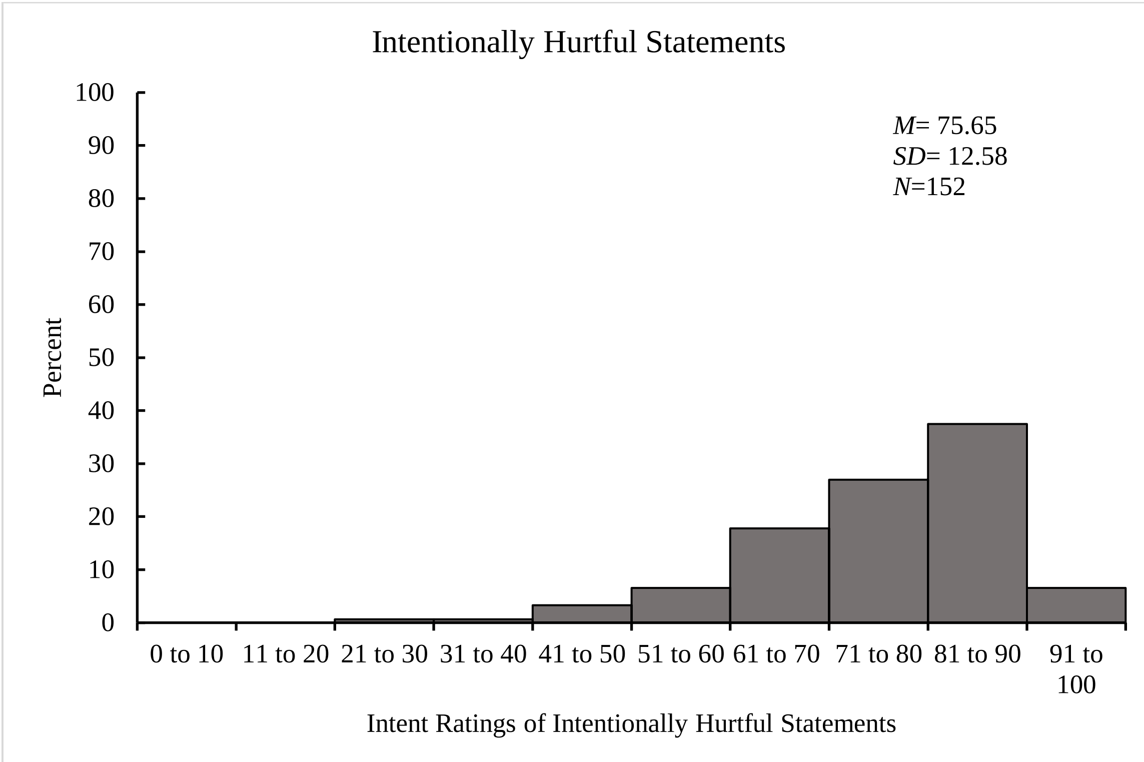

Figure 3.2: Intent ratings in response to intentionally hurtful statements.

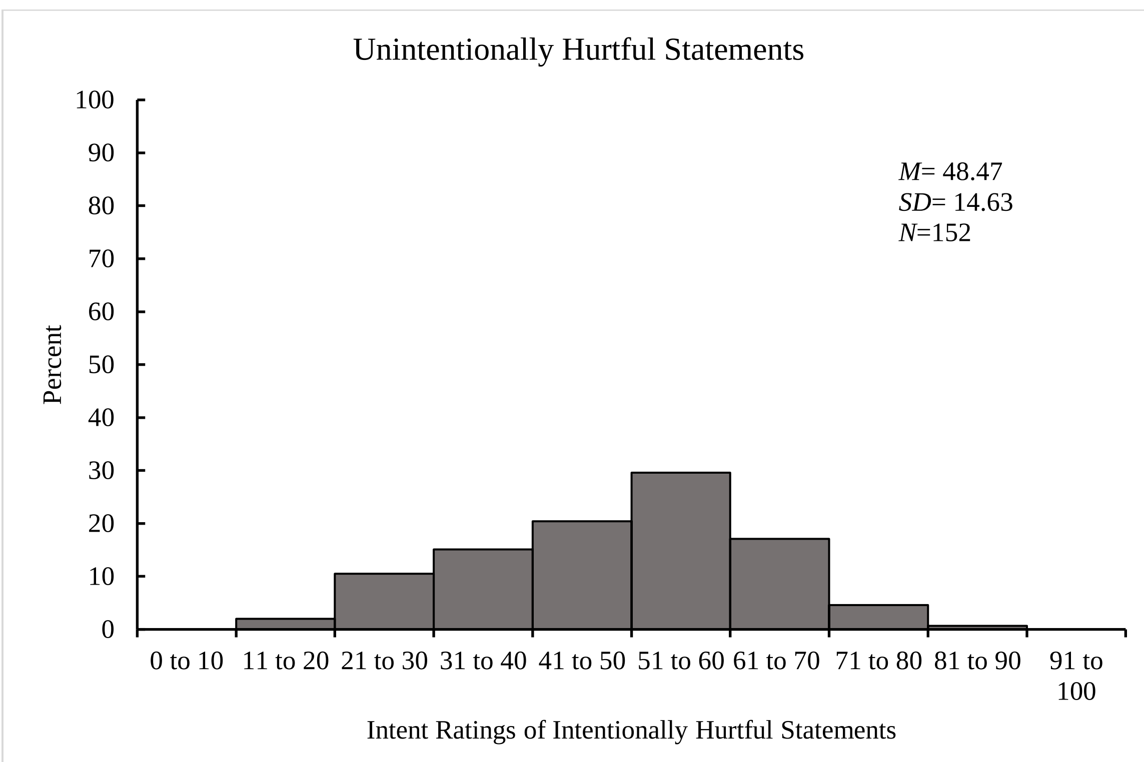

Figure 3.3: Intent ratings in response to unintentionally hurtful statements.

Table 4

	Positive Statements	**Intentionally Hurtful Statements**	**Unintentionally Hurtful Statements**
Stress Reaction	r=.11 95% CI [-.05, .26]	r=.11 95% CI [-.05,.27]	r=.08 95% CI [-.08, .24]
Alienation	r=.20* 95% CI [.04, .35]	r=.06 95% CI [-.10, .22]	r=.12 95% CI [-.04, .28]
Words can Harm	r=.02 95% CI [-.16, .19]	r=.35*** 95% CI [.19, .49]	r=.35*** 95% CI [.19, .49]

Note. * $p < .05$, ** $p < .01$, *** $p < .001$. *Correlations between participant's perceptions of the intent of the person say the statement on each of the three different statements and their response to the negative emotionality facets and the words can harm scale.*

Discussion

We designed this study to determine whether there is any consensus about what other people perceive as harmful when it comes to the things that people say to one another. We found, as expected, clear consensus that *positive* statements are perceived as not hurtful or intended to be hurtful. However, we found far less consensus in participants' reactions to intentionally and unintentionally hurtful statements. Our findings reinforce Lilienfeld's suggestion that the microaggression construct is, at this point, ambiguously defined. That is, it is an open concept, and until it can be measured with tools demonstrating both reliability and validity, we should be using it with caution. Thus, before administrators identify a statement as harmful, we suggest they provide evidence that individuals across a wide span of contexts and individual identities perceive it as harmful.

A second objective of this research was to determine whether individual differences in personality traits are linked to how people react to others' words. Indeed, our participants showed strong internal consistency in their ratings from one statement to the next, such that we were justified in aggregating each participant's responses to the 16 positive statements, the 16 intentionally hurtful statements, and the 16 unintentionally hurtful statements. Further, participants' aggregate scores were, indeed, related to their level of negative emotionality (neuroticism). People higher in stress reaction (trait anxiety) consistently reacted more negatively to the hurtful statements, whether they had been nominated as intentionally hurtful or not... Although some things people say are undoubtedly hurtful, our findings imply that there is also the complicating factor that some people are more likely than others to perceive relatively innocuous statements as hurtful. Notably, we specifically used statements in our study that stayed away from group identity variables such as gender, race, sexual orientation, and class, and thus we perhaps gave ourselves a conservative test of the possibility that individual differences would relate to people's reactions to the statements. If "microaggressions" are subtle insults and snubs, some people may be far more likely than others to perceive themselves as being victims of microaggressions, no matter whether the context is one tied to group identity or not. As others have noted, people who perceive themselves as microaggressed against also are more likely to struggle with mental health concerns like depression and anxiety. Although the data are observational, one common reason offered for link is that microaggressions *cause* mental health issues. Our data raise the alternative possibility that negative emotionality is a third variable that may lead to both (1) perceiving oneself as receiving microaggressions and (2) struggling with mental health. Plenty of studies have documented ties between negative emotionality and psychological well-being; the current data have just provided documentation of the tie between negative emotionality and reacting negatively to statements nominated precisely because they were not meant to harm. In conclusion, given the lack of clear consensus about how hurtful various statements are, and the link between negative emotionality and perceiving statements as hurtful, we encourage systematic research that will carefully work toward operationalizing and testing the microaggressions construct. In the meantime, we worry that delivering (mandatory) microaggressions workshops across the academic and corporate world may have unintended consequences. Microaggression workshops have been incorporated into regular business without rigorous testing of their consequences. We await randomized controlled studies that randomly assign people to attend (or not attend) a microaggressions workshop, and then compare workshop attendees' and non-attendees' attitudes about the power of words to be hurtful and cause harm. If attendees in such studies perceive words as more powerful and hurtful after the workshop than non-attendees do, what would that mean, exactly? And how would that have increased perception of words as harmful play out in their subsequent outlook on other people and in their behavior and interactions with others? In the context of our data, we worry that the explosion of the microaggressions construct may have some unintended iatrogenic effects.

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Evaluating Video Journals as a Tool for Home Voice Practice of Vocal Hygiene

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Abstract

Extensive vocal hygiene behavior changes can be challenging for patients to achieve without accountability outside of treatment. Frequently clinicians rely upon patient report or voice journaling to track vocal hygiene behaviors between sessions. Use of technology, such as smartphone applications, has been shown to be effective in adherence to other voice exercises (van Leer & Porcaro, 2018). The use of video or audio journals to record vocal hygiene behaviors and complete self-assessments of related voice quality may be helpful for patients. This study compared two different modes of journaling, video journals and paper journals, to a no journal control group. Outcome measures included use of journal, use of vocal hygiene behaviors, changes to voice function, and participant perceptions.

Literature Review

Speech language pathologists, SLPs, are professionals that serve patients and their families/caregivers in both medical and educational settings. ASHA, the American Speech- Language Hearing Association, states: "Speech Language Pathologists work to prevent, assess, diagnosis and treat speech, language, social communication, cognitive communication, and swallowing disorders in children and adults" (American Speech-Language Hearing Association, n.d.). SLPs work to treat disorders, alleviate symptoms, provide various supports, and help families/caregivers in understanding and coping with diagnoses that affect cognition, speech, language, or swallowing. The focus of the current study is the area of voice disorders.

The larynx, also known as the voice box, is located above the windpipe in the neck. The vocal folds are two pieces of tissue that are about at the level of the Adam's Apple. The vocal folds are attached at the front and can open and close at the back. When breathing, they are in an open position. During phonation (talking), closure and vibration of the vocal folds occurs. The vibration is due to the airflow that is provided by the respiratory system. When all of these systems (respiratory, larynx, and upper airway of the pharynx, oral cavity, and nasal cavity) work appropriately, typical voice and speech occurs.

A voice disorder is diagnosed when an individual expresses concern about challenges relating to an abnormal voice, which does not allow him/her the ability to meet every-day needs (American Speech-Language Hearing Association, n.d.). Individuals with voice disorders can experience loss of voice, pain, difficulty with effective communication for everyday personal and professional interactions, fatigue, tension, and a myriad of other symptoms. Voice disorders can arise from organic sources such as nodules or polyps, neurological sources such as tremor or paralysis, or vocal abuse and/or misuse.

Hyperfunctional voice disorders are disorders seen when an organic or neurological vocal fold pathology cannot be identified. These can occur due to vocal misuse or abuse from excessive use, intensity of speaking, laryngeal tension, muscle compensation, and psychological factors (Oates, J., & Winkworth, A. 2008). Muscle Tension Dysphonia (MTD) is a common example of a hyperfunctional voice disorder in which the tension can prevent the voice from functioning properly. Tension can cause pain, fatigue, weakness, or voice change.

When seeking aid from an SLP, an assessment is conducted during the initial appointment to provide the SLP with adequate information to plan an appropriate direction for treatment. Components of the assessment conducted are clinician- client interview, where the SLP will gain knowledge about the symptoms the client is experiencing, the onset and duration of the problem, associated symptoms, and typical voice use. After that, perceptual and acoustic voice assessments and direct or indirect visualization of the vocal folds will also be performed. Information learned from the intake appointment aids in being able to properly choose follow-up assessments that will be beneficial in finding a diagnosis or to make a referral to a laryngologist (Ear, Nose, and Throat doctor, n.d.).

The clinical interview is typically the first step in any evaluation. It provides detailed information from the client's perspective about voice function. Clinicians may use rating scales, like the Voice Handicap Index (VHI) (Jacobson et al, 1997) or Voice Related Quality of Life (V-RQOL) (Hogikyan & Sethuraman) to gain an objective assessment of how the voice impacts the client's daily life.

After completing the clinical interview, the clinician will do a perceptual evaluation. This involves careful listening to the various aspects of the client's voice and documenting severity using a scale, such as the Consensus Auditory Perceptual Evaluation of

Voice (CAPE-V) (Kempster et al., 2009) or Grade Roughness Breathiness Asthenia Strain Scale (GRBAS) (Hirano, 1981). Clients perform a variety of tasks like sustained phonation, reading sentences, and providing spontaneous speech samples.

Acoustic or aerodynamic measurements reflect the sounds and airflow energies as the vocal folds vibrate. This includes fundamental frequency, which is the acoustic measure that reflects the rate of vibration of the vocal folds, measured from a sustained vowel. Other measures, like jitter, shimmer, and harmonic to noise ratio help quantify the periodicity of vibration and extent of closure, both of which are needed for good voice quality. Phonation range, the frequencies from the highest to lowest that the patient is able to produce, is another useful measure taken to document the client's ability to stretch and contract the vocal folds. Dynamic range tasks require the client to produce the softest /ah/ and then produce the loudest /ah/ possible to assess the coordination of the respiratory and laryngeal function as well as the size of the vocal fold vibration. The clinician may have the patient produce a maximum phonation time task, which requires the patient to hold out a vowel sound /ah/ at a comfortable pitch for as long as possible. This task assesses both respiratory support and the efficiency of the vocal fold vibration.

Visual assessment of the vocal folds includes directly observing the vocal folds through oral or nasal endoscopes or indirectly using a mirror exam. Oral or nasal endoscopy includes using a camera attached to a scope that gives the ability to see the vocal folds in motion. This is important as SLPs can look at the closure pattern of the vocal folds, vibration of the folds, any growths/lesions, color of the structures, squeezing of the structures, etc. Results of the visualization may guide treatment planning to change anatomy or physiology of the larynx as needed.

Treatment can vary based on the type of the voice disorder. Treatment options range from implementation and change of behaviors for better quality voice to medical and/or surgical treatments. Surgery is the most invasive treatment as it involves cutting into tissues, which can alter the tissues and the shape or position of the structures surrounding the vocal folds. Surgery is usually chosen in cases to remove lesions, modify the position or shape of the vocal fold, or even inject and implant materials. "The goals of surgical management are to conserve, reconstruct, or improve laryngeal functions—phonation, swallowing, and respiration" (Colton, R. H., Casper, J. K., & Leonard, R. (2012).

Voice therapy may be determined to be the best option for treatment. Speech-language pathologists are often called upon to provide treatment to regain healthy voice and adequate function. SLPs use different approaches for treatment, which can be classified as direct treatment and indirect treatment. Direct treatment is an option that attempts to directly change the vocal mechanism, while indirect treatment modifies cognitive, behavioral, psychological, and physical environments.

Direct treatment techniques, like resonant voice therapy (Resonance Disorders), Casper Stone flow phonation (American Speech-Language Association, n.d.), circumlaryngeal massage, or muscle tension reduction exercises, require the client to alter the function of the vocal mechanism. Resonant voice therapy and Casper Stone flow phonation require the individual to produce oral vibratory sensations in the face for easier vibration of the vocal folds; they are designed to directly change the amount of adduction of the vocal folds. Circumlaryngeal massage and muscle tension reduction exercises are techniques that are used to reduce the excessive tension in the neck and laryngeal muscles; this can directly alter the type and amount of squeezing of the vocal folds for phonation. Many other direct treatments exist.

Indirect therapy techniques, like altering the environment in which the client speaks, require the client to change aspects of how the voice is used. In cases where a voice disorder develops due to the way a person uses his/her voice (or misuses his/her voice), treatment by an SLP often requires training the patient in healthy vocal behaviors. Vocal hygiene is more specifically a habit that supports a healthy and strong voice. Some healthy vocal hygiene habits include increased hydration, avoiding alcohol and smoking, reduced intake of spicy foods, awareness of throat clearing, and controlling loudness (Ear, Nose, Throat Consultants, n.d.). Permanent damage to the voice is possible with prolonged poor vocal hygiene.

Although there are many aspects of vocal hygiene, this study focused on three habits: reduction of loud talking and yelling, reduction of throat clearing, and vocal rest. These are important to vocal hygiene as loud talking and yelling or throat clearing can cause trauma or strain to the vocal folds to the vocal folds. Vocal rest is also important as it allows the tissues to rest from the vibration created due to talking. Constant use and especially misuse could cause vocal strain, excessive tension, and even vocal nodules.

Previous research has suggested that vocal hygiene may contribute to reduced perception of vocal handicap (Behrman, Rutledge, Hembree, & Sheridan, 2008), improved acoustic measures of voice function (Chan, 1994), improved voice quality (Verdolini-Marston, et al., 1994), and improved measures of effort in voicing (Verdolini-Marston, et al., 1994). However, training in vocal hygiene, without daily behavior tracking, does not result in changes to most voice measures (Broaddus-Lawrence, et al., 2000). In one study, individuals who adhered to a regimen of vocal hygiene behavior showed significant improvements in perceived voice function as compared to those who reported that they did not adhere to the treatment plan (Behrman, Rutledge, Hembree, & Sheridan, 2008).

Patients may find it challenging to comply with an extensive list of recommendations and behavior changes, specifically when no accountability is required (Broaddus-Lawrence, et al., 2000). Current practice within speech-language pathology utilizes patient self-report about the adherence to vocal hygiene recommendations, either with or without written supports such as a voice journal (Broaddus-Lawrence, et al., 2000; Chan, 1994). Use of technology, such as smartphone applications, has been proposed and shown to be effective in adherence to other voice exercises (van Leer & Porcaro, 2018). However, to date, no studies have examined the use of voice or audio journals to record self-assessments of adherence to vocal hygiene recommendations. The immediacy, effectiveness, and reflective nature of video recordings may contribute to better adherence and overall improvements (Aitken & Deaker, 2007). This study will examine the benefits to different modes of journaling and find whether or not it increases patient compliance to vocal hygiene behaviors.

The following research questions were developed:

1. Do participants use paper and video journals for documenting vocal hygiene with similar compliance?
2. Does a video journal help participants use positive vocal hygiene habits, as compared to paper or no journal use?
3. Does a video journal improve voice function, when using positive vocal hygiene habits, as compared to paper or no journal use?
4. Do participants rate their voice function differently after receiving vocal hygiene education and opportunities to practice?

Methods

Participants

Eleven students from the University of Wisconsin Eau Claire College of Education and Human Sciences were recruited to participate. The participants averaged 20.9 years (range 19-29 years). All participants were female; gender was not an

Table 1
Participant demographics

Participant	Age	Stress Score (max of 5)	Daily Water Intake (avg. ounces)	Daily Caffeine Intake (avg. ounces)	Daily Voice Usage
P1	21	3.5	60	20	Loud talking at work and typical conv.
P2	20	4	16	8	Minimal; yelling at volleyball
P3	20	3	40	10	Typical conversation, singing, yelling
P4	21	3	60	4	School and social
Paper group average	20.5	3.4	44	10.5	
NJ1	21	3.5	60	0	Social, athletics and phone talking
NJ2	20	2	40	16	Social and work
NJ3	20	2	40	14	Work, school/educating
No Journal group average	20.3	2.5	46.7	10	
V1	19	2	25	0	Conversational
V2	21	4	80	4	Conversational
V3	18	3	32	16	Work and social
V4	29	2	40	24	Work related
Video group average	21.8	2.8	44.3	11	
Overall	20.9	2.9	44.8	10.6	

inclusion/exclusion criterion. All participants reported no history of voice or speech disorders prior to beginning the study. The study was approved by the IRB at UW Eau Claire and all participants provided informed consent prior to beginning participation. Participants within each group were similar across measures.

Procedure

Participants were randomly assigned to one of three groups: 1) no journaling, where the participant reported vocal hygiene behaviors weekly during a meeting with the primary investigator; 2) paper journaling, where the participant recorded vocal hygiene behaviors daily on a paper form and presented these to the PI weekly (Appendix A); 3) video journaling, where the participant recorded short "selfie" videos daily about vocal hygiene behaviors and presented these to the PI weekly. Each participant completed four visits. The first visit was an initial baseline evaluation. The second and third visits were meetings with the PI to share vocal hygiene behavior use; those in groups using journaling presented their journals at this time. The fourth visit was a final sharing of vocal hygiene behaviors, as well as a final voice evaluation.

Participants initially completed a brief evaluation session. Following informed consent documentation, participants completed a demographic questionnaire to get information about gender, age, voice history, and estimated vocal hygiene behavior implementation. The Voice Handicap Index was then administered to each participant. Recordings of the participants' voices were completed in a sound-proof booth, using the Marantz Solid State Recorder PMD661 MK II. Participants provided sustained phonation, sustained /s/, sustained /z/, glides up and down, sentences from the CAPE-V, the Rainbow passage, and a 30-second monologue. Next, each participant received a stroboscopic exam to visualize the larynx, specifically the structure and vocal fold vibration. During the exam, participants completed the following: sustained phonation, gliding up and down, loud and soft phonation, and a laryngeal diadochokinesis task. Following the voice measures and stroboscopic exam, education was provided related to vocal hygiene, with special discussion of the three behaviors to be tracked in this study: 1) reduction in loud talking/yelling; 2) reduction in throat clearing; 3) implementation of vocal rest. Finally, participants were trained in their assigned type of journaling (none, paper, or video).

Each week, participants focused on vocal hygiene habits and recorded data as assigned. Participants assigned to the no journal group were asked to be aware of their habits, but not to track any specific behaviors throughout the week. Participants assigned to the paper journal group were provided a folder with pre-typed questions to answer on a daily basis related to specific vocal hygiene behaviors. Participants assigned to the video journal group were provided an Apple 6th generation iPod to record brief video reflections on a daily basis related to specific vocal hygiene behaviors. Questions were provided for making their videos. Journal questions are provided in (Appendix A).

On a weekly basis, participants met with the PI and provided a brief reflection of the week overall, in terms of vocal hygiene use. Specifically, participants were asked to report use of intentional vocal rest, frequency of throat clearing, and frequency of loud talking/yelling on a three-point scale (none, some, lots). Then, participants were asked to report whether their awareness had changed, how their voice sounded, and whether the habits were affecting their voice. These responses were recorded by the PI; an example of this questionnaire is available in (Appendix C). The PI also collected the daily journals from the past week from the participants in the paper and video groups. At their final visit, each participant completed a follow-up questionnaire to reflect on their experience and then repeated the voice recordings and stroboscopic exam used in the baseline evaluation.

Data Analysis

Audio files were uploaded from the Marantz Solid State Recorder PMD661 MK II into the MDVP Advanced Version on the Model 4150B Computerized Speech Lab (CSL). Sustained phonation samples were analyzed first, gathering F0, MaxF0, Min F0, SD F0, using the single token protocol for each phonation sample. Statistical data were listed in the MDVP Report: Voice Report. This report generated Average Fundamental frequency (mean F0), Jitter (RAP), Shimmer (APQ), and Noise to Harmonic Ratio (NHR). S to z ratio was calculated by timing the duration of the sustained /s/ and the sustained /z/ using the CSL. Next connected speech samples, including the CAPE-V Sentences, the Rainbow Passage, and a thirty-second monologue were analyzed under the single token protocol for fundamental frequency, phonatory F0 range (PFR), maximum and minimum fundamental frequency (fhi and flo), and also the Cepstral Peak Prominence (mean and standard deviation). Glides were analyzed for high and low frequency using the CSL.

Each file was saved as a wav audio file under the assigned number. For overall perceptual rating of the monologue, rainbow passage, sustained phonation, and CAPE-V sentences, the files were assembled into a playlist using PowerPoint with each participant using their assigned number. Each playlist contained both pre- and post-journaling files and repeated eight files for intra-rater reliability calculations. Two researchers rated the overall severity and noted any roughness, breathiness, or strain in each sample. The overall rating was on a 0 to 5 scale, with 0 being normal and 5 being severely abnormal. These ratings were selected to make a more precise decision compared to the typical 3-point, mild, moderate, severe scale. An example of the rating form is included in (Appendix D). The two listeners' ratings were averaged for each sample. Comparisons were made across groups and across individuals to determine whether change occurred over the course of the study.

Stroboscopic exams were viewed by the same two researchers to score closure pattern, vocal fold edge, supraglottic involvement, amplitude of vibration, and mucosal wave. These are frequently measured characteristics of vocal fold movement and vibration during stroboscopic exams. Each characteristic was rated from 0-5, with zero showing none/normal and 5 showing severely abnormal function. An example of the rating form is included in (Appendix E). Change scores from pre to post were calculated, and comparisons were made within and across groups.

The VHI forms were scored for each subscale (i.e., physical, functional, emotional), as well as the total for each participant for the pre- and post-study assessments. Group means were calculated for comparison across the journal types.

Each participant's journal entries were recorded by date. The number of entries was counted and converted to a percentage based on the number of possible entries throughout the course of the study. Journal completion was compared between video and paper groups.

Vocal hygiene behaviors were also tracked based on journal entries submitted to the PI during the weekly meetings. Daily questions about voice rest, throat clearing, and loud talking were assigned a score and reported across the duration of the study for the paper and video groups. Weekly meeting responses related to voice rest, throat clearing behavior and awareness, and loud talking behavior and awareness were assigned a score and reported across the study's duration. Comparisons across groups were completed with a two-way ANOVA.

Results

Journal Compliance

Each participant in the paper and video groups had an opportunity to journal every day from their first to the last session (n=20-25 days; variability based on start and end dates for some participants). The paper journal group averaged a 95.6% completion rate while the video journal group averaged an 89.3% completion rate (Figure 1). Three participants had 100% completion (two from the paper group and one from the video group). Of the remaining five participants, three had greater than 90% completion rates. Only one participant in each group had less than 90% completion.

Open-ended responses to questions about journal type and effectiveness during the participants' final session were recorded. Those who were in a no journal group felt that they had gained more awareness in vocal hygiene but lacked something to hold them accountable for implementing the habits. Participants in the paper journal group stated that their journal was easy to use due to the pre-typed questions. The physical journal served as a reminder to implement vocal hygiene practices and journal their experience. The video journaling group also felt that the journal was easy to use, and it was an efficient way to record their experience. Similar to the paper group, the video group emphasized that the iPod served as a reminder to implement vocal hygiene habits and journal their experience. Neither group identified any major concerns with their journaling type, although one participant in the video journal group reported that recording the videos made her nervous.

Figure 1
Journal Completetion by Group

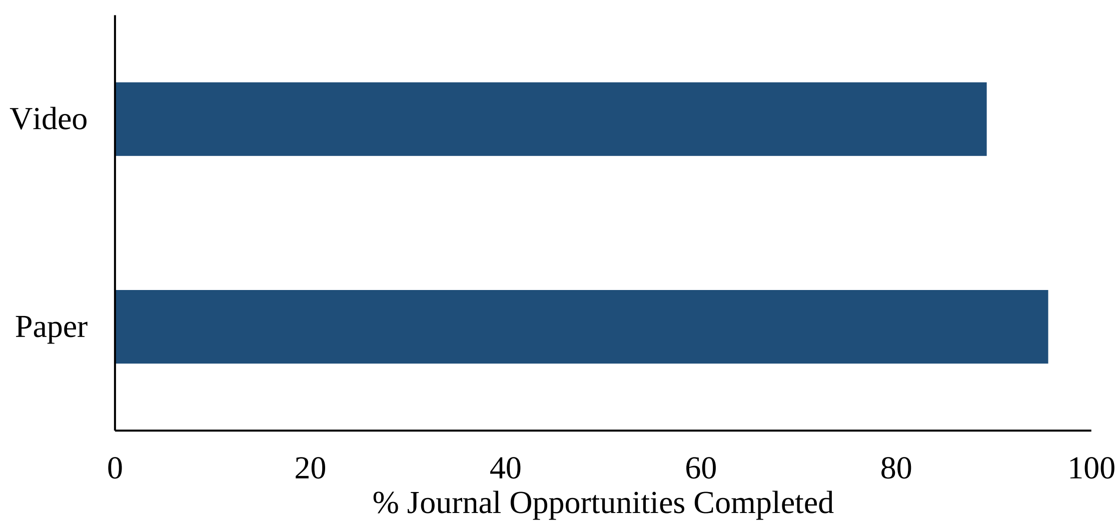
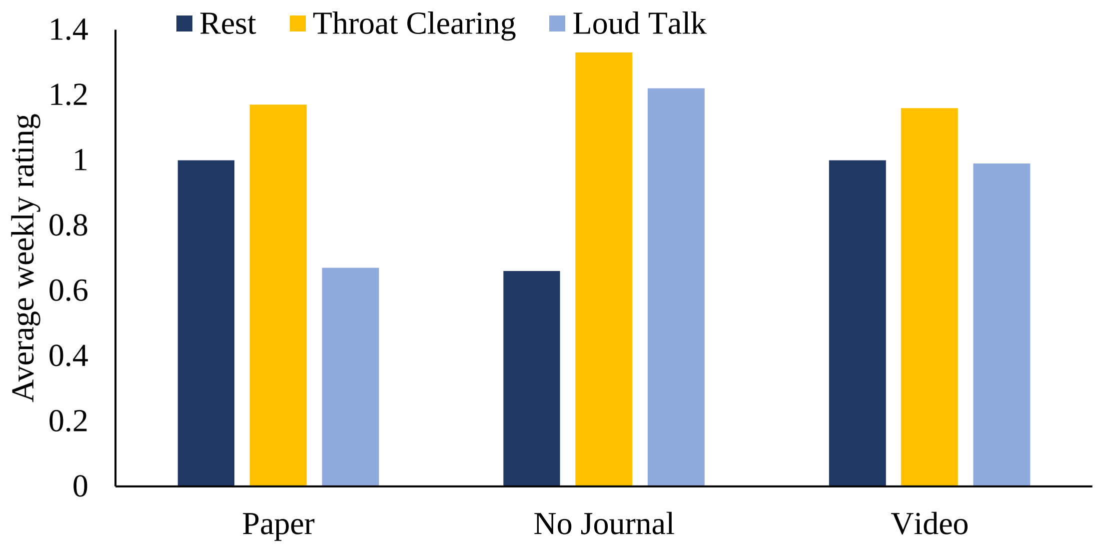


Figure 2

Vocal Hygiene Behavior Usage Between Groups



Use of Vocal Hygiene Habits

Figure 2 provides average vocal hygiene usage per week across each group. For the behavior of vocal rest, participants in the paper and video journal groups more consistently used this behavior than the no journal group. No statistically significant differences between groups, on the basis of vocal hygiene behavior usage, were found (two-way ANOVA for group and behaviors using Roy's Largest Root, F=1.37, p=0.329).

Further analysis of the day-to-day use of habits for participants in the video and paper groups was completed by plotting ratings of habit throughout the duration of the study. More participants in the video group reported using the habit of reduced loud talking on a consistent basis (Figure 3); in the paper group, a larger number of participant responses indicated that they did not implement the habit of reduced loud talking at all (Figure 4). Only one participant in the video group did not implement the habit and that only occurred several times. In the paper group, three of the four participants had instances of not implementing the habit, with two of those having frequent days without implementation. Similar results were noted in the habit of throat clearing. In the video group, one participant reported not using the habit, and instances of occasionally performing loud talking were documented. (Figure 5). In the paper group, all four participants reported at least one instance of using throat clearing, although for this habit, most only reported use of the habit on 1-2 days (Figure 6).

Figure 3.

Use of Loud Talking (Talk) Over Time by Participants in the Paper Journal Group.

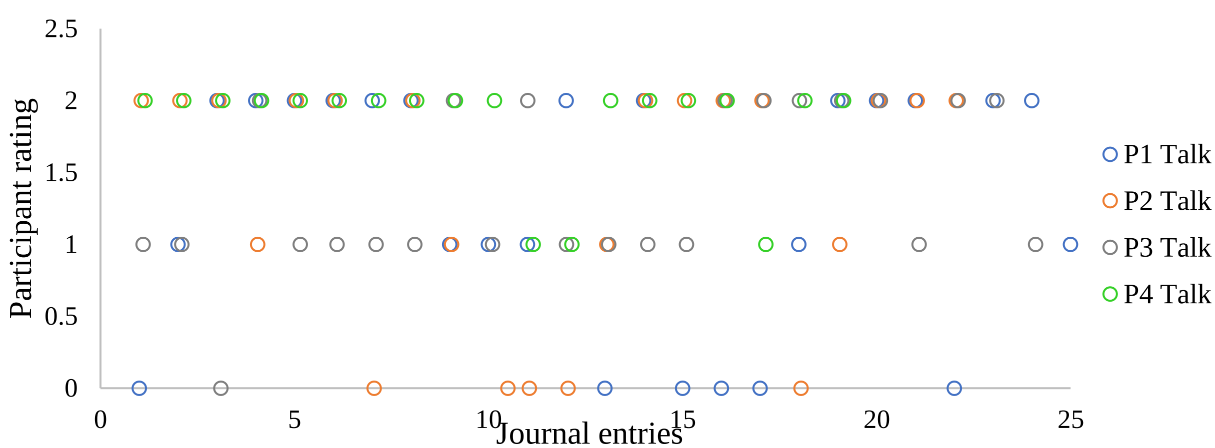


Figure 4.
Use of Loud Talking Over Time by Participants in the Video Journal Group.

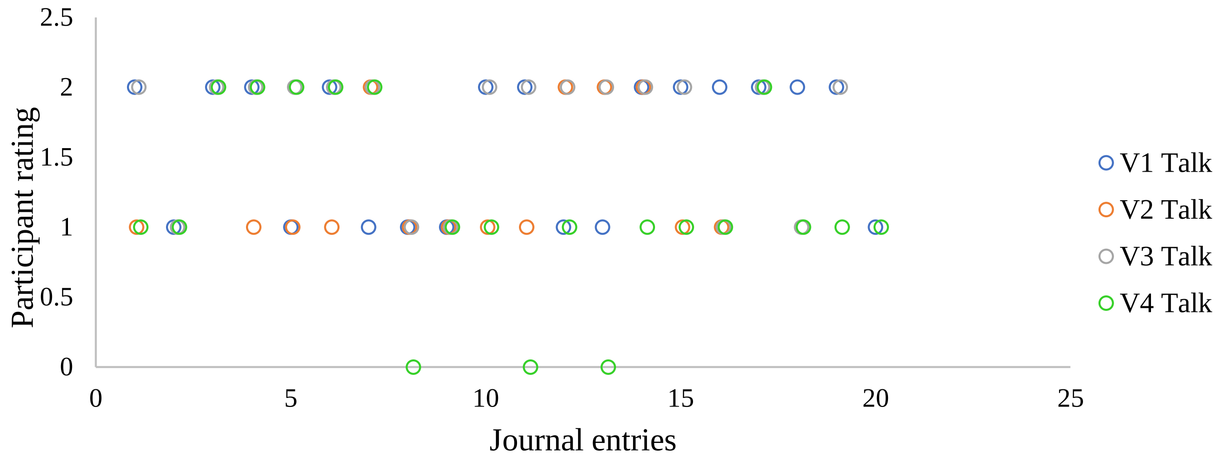


Figure 5.
Use of Throat Clearing (TC) Over Time by Participants in the Paper Journal Group.

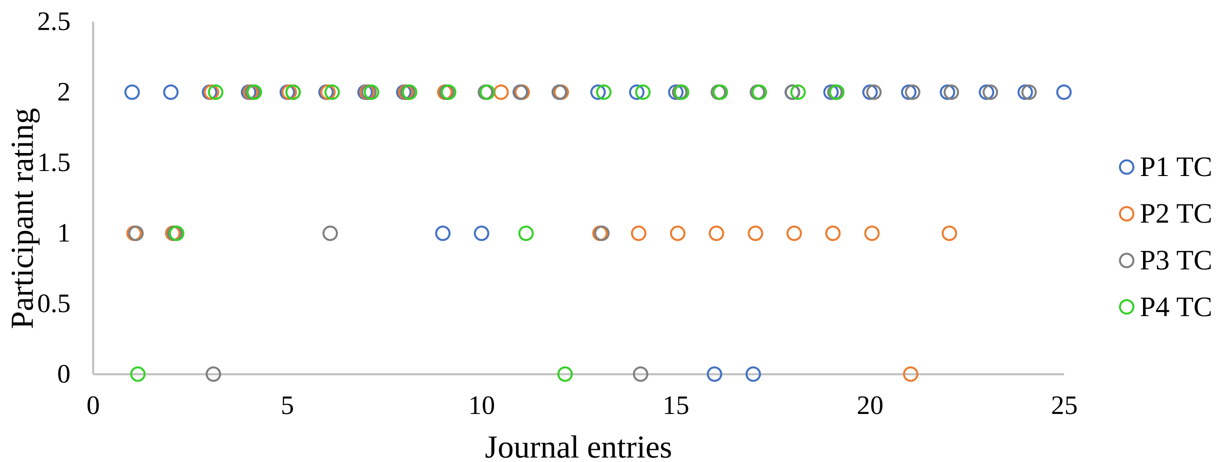


Figure 6.
Use of Throat Clearing Over Time by Participants in the Video Journal Group.

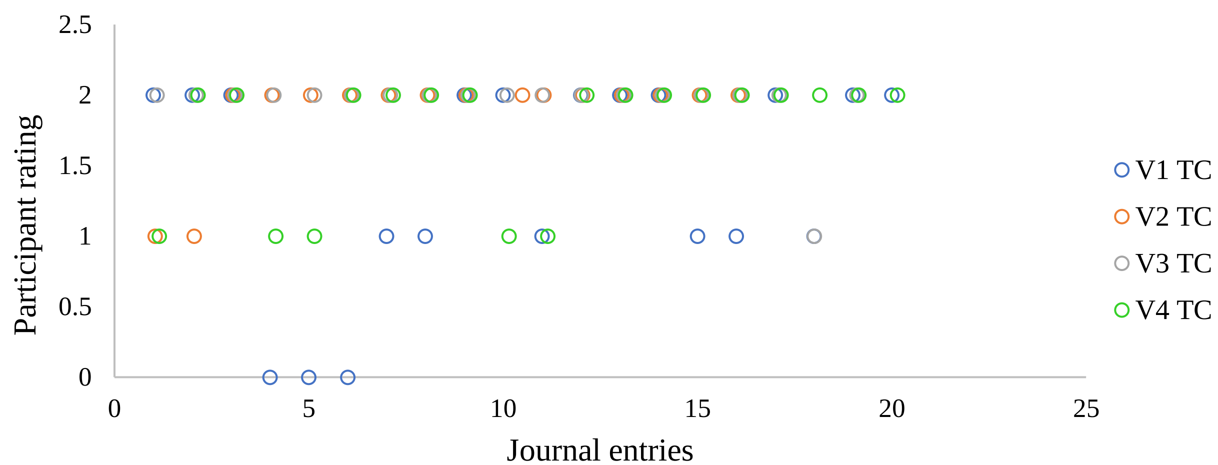
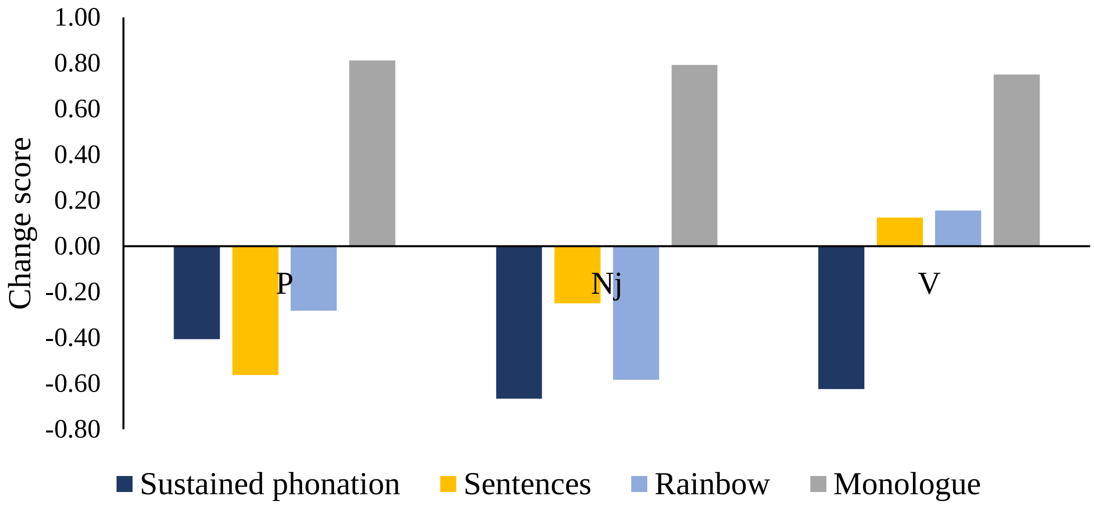


Figure 7

Mean Perceptual Voice Rating Change Scores for All Groups



Overall, paper and video journal groups provided more consistent responses during their weekly check-ins to questions like: "did you feel more aware of your loud talking or throat clearing" and "did you intentionally rest your voice." Although these groups were able to answer questions with greater confidence, all groups reported an increased awareness of vocal hygiene adherence. All groups reported implementation of these habits into their lives, but those without a journal reported that there was less of a change.

Vocal Function Changes

Figure 7 shows perceptual change scores across three different tasks; change scores were calculated by subtracting the pre-therapy rating from the post-therapy. For the perceptual ratings, a score of 1 indicated normal voicing while a score of 5 indicated severely impaired voicing; averages of the two judges were used. Thus, change scores that are negative are considered better or showing improvement. The scores ranged between one and two, as all participants enrolled inthe study had self-reported normal voice function. All participants showed better vocal quality on sustained phonation at post-test; conversely, all participants showed poorer vocal quality at the post-test on the monologue.

Table 2 provides acoustic data as change scores for each group. For jitter, shimmer, and NHR, negative change scores indicate an improvement from pre-test to post-test. For PFR and CPP, positive change scores indicate an improvement from pre-test to post-test. For ease of reading Table 2, improved scores have been highlighted. Change scores, for all measures, are extremely small.

Ratings from the physical examination of the larynx using stroboscopy showed very little difference between pre-test and post-test. Scores range from 0 (normal) to 1 (slightly abnormal) across all five areas assessed. Most ratings, when averaged between the judges, were scored as zero at the pre-test and post-test. The stroboscopic exam showed that all participants were normal to start.

Quality of Life/Rating of Voice Function

The participants completed the Voice Handicap Index at the first and last session. Most individuals rated their voices better at post-test (see Figure 8). One participant in the video group had a slight increase in their rating.

Table 2

Mean acoustic data for each group

Group	Jitter	Shim.	NHR	PFR sent.	CPP sent.	PFR Rain.	CPP Rain.	PFR mono.	CPP mono.
No journal	-0.15	-0.70	-0.01	-3.00	0.27	-1.33	-0.20	1.67	0.54
Paper	0.02	-0.47	0	-1.50	-0.05	1.00	-0.24	-5.00	-0.03
Video	0.12	-0.14	0	3.50	-0.49	-1.75	0.21	-0.50	0

(Shim. = shimmer; sent. = sentences; Rain. = Rainbow passage; mono. = monologue)

Figure 8

Mean Average Rating of Voice Handicap Index for All Groups

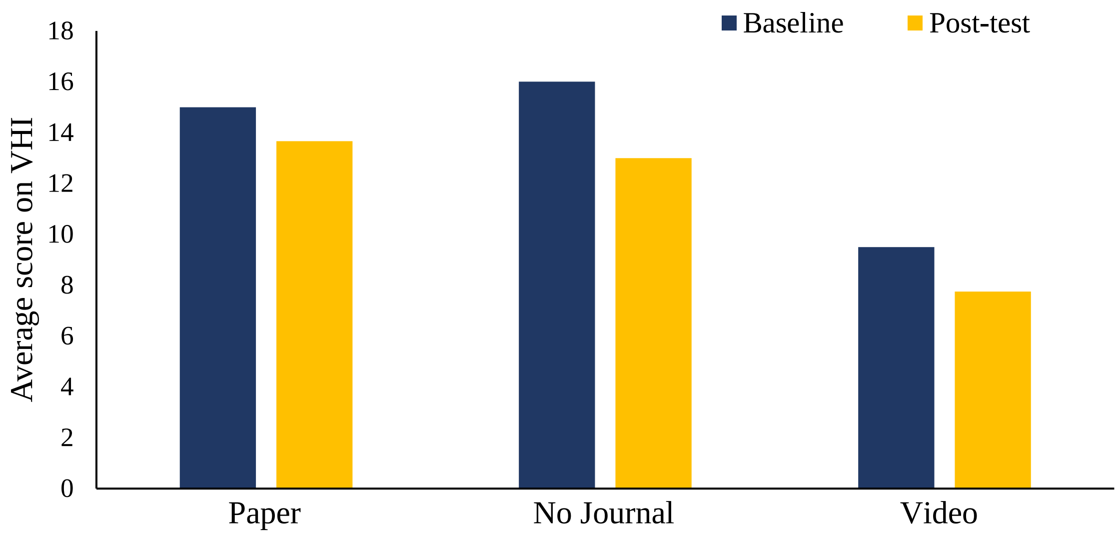


Table 3

Inter-Rater and Intra-Rater Reliability for Perceptual and Stroboscopic Analyses

	Sustained Phonation	CAPE-V Sentences	Rainbow Passage	Monologue	Strobe
Inter-rater reliability (% agreement)	100%	100%	100%	97%	89%
Intra-rater reliability					
Judge 1 (AH)	86%	100%	100%	100%	100%
Judge 2 (KC)	86%	100%	100%	100%	100%

*Judged to be in agreement if within at least one point

Reliability

Table 3 shows percentages for both inter-rater and intra-rater reliability. The raters had complete agreement across most tasks, with the stroboscopic ratings showing the most differences. Within each judge, agreement was very high across tasks. Both inter- and intra-rater reliability were deemed acceptable.

Discussion

Do Participants Use Paper and Video Journals for Documenting Vocal Hygiene with Similar Compliance?

Participants assigned to the video group had an 89.3% completion rate while participants in the paper group had 95.6% completion rate. Both groups had good compliance with journal completion, regardless of method of journaling. The video group completion rate was influenced by a single participant who missed six recordings (of a required 21). It is possible that this participant completed the activities and reflections, but the technology failed to record it. A second possibility is that she simply didn't have time to do all of the video journals. This participant did not provide an explanation at the final interview. Overall, participants in the video group, other than this single participant, completed the journals with equal compliance to the paper journal group.

Participants reported that paper journals or the video journals were helpful for increasing compliance. In addition, the video journal group had the ability to set a reminder for whenever they desired to complete the journal. Some participants also entered a reminder in their personal planner, which may explain why those in the paper journal group had high levels of compliance. Participants who were in the no journal group did not use a personal planner (they were asked not to) and did not have scheduled time to reflect on their vocal hygiene. Not surprisingly, they shared in the follow-up that it was difficult to remember to practice because there was nothing to hold them accountable.

Does a Video Journal Help Participants Use Positive Vocal Hygiene Habits, as Compared to Paper or No Journal Use?

During the weekly meetings, participants reported estimated use of vocal hygiene habits over the previous week. No statistically significant differences between groups, on the basis of vocal hygiene behavior usage, were found. In other words, all groups were implementing vocal hygiene behaviors to some extent. The initial session with all participants was an educational session about vocal hygiene. In addition to learning about what vocal hygiene is and the three habits this study addressed; the participants also learned about what can happen when poor vocal hygiene is used. It is possible that this education primed them to begin using vocal hygiene habits more regularly, similar to the findings of Broaddus-Lawrence and colleagues (2000), who found their participants implemented their new knowledge about vocal hygiene.

The journals were an aid in holding participants adherent to changes in vocal hygiene behaviors, similar to Broaddus-Lawrence and colleagues (2000). At the final session, participants were asked about their thoughts about the assigned journal modality they had. Both the paper and video groups mentioned that having a physical journal had reminded them to complete their entries, while those in the no journal group reported some difficulty remembering to implement behaviors consistently. It was hypothesized that the video journals would see a higher rate of completion and better use of vocal hygiene habits due to videos having more immediacy and being reflective in nature, which could contribute to implementation of habits and overall better adherence (Aitken & Deaker, 2007). Within this study, paper journals and video journals seemed to both serve this purpose.

Participants' implementation of the vocal hygiene targets of reduced throat clearing and reduced loud talking were assessed on a daily basis within each individual throughout the duration of the study for those in groups with some sort of journal. The participants in the paper group implemented habits less frequently than those in the video group, although both groups did implement behaviors relatively consistently. Both groups used reminders, but those in the video group, had electronic reminders with an audio component, which may have served as a more alerting reminder to complete vocal hygiene tasks.

All video journal group participants received an iPod, on loan, to complete their recordings. Specific to the video journaling, it was hypothesized that participants would utilize the video method more eagerly. The participants in this study use technology on a daily basis, for personal and academic purposes. Participants reported that the video journal was easy to use and conformed to their lifestyle. Van Leer & Porcaro (2018), using a fake phone call protocol, illustrated that video/audio journaling or therapy practice could be effective, similar to the themes found in our use of video journaling. The video group did see a lower completion rate than the paper. It is possible that the addition of a second device, the iPod, in addition to their own devices, created an extra layer of difficulty in remembering to access the device. Perhaps an application on their own devices would make this intervention more effective.

The participants in the video group provided more detailed responses in their daily reflections. This could be due to the ease of using video journals for recording one's thoughts. With video journals, participants were able to say exactly what they wanted without having to put in the effort of writing their thoughts. Paper group participants may have restricted themselves to writing brief thoughts, which could have affected thinking and implementing positive vocal hygiene behaviors.

Does a Video Journal Improve Voice Function, When Using Positive Vocal Hygiene Habits, as Compared to Paper or No Journal Use?

Participants all had an improvement in their voice. Specifically, perceptual measures increased across all groups. Acoustic measures had a slight improvement, and the stroboscopic results remained consistent from pre-test to post-test. The changes in all measures were very small in nature, which is consistent with day to day fluctuations in voice function. However, participants reported feeling their voice improved when they implemented vocal hygiene. Implementing healthy behaviors could lead to positive thoughts and feelings about their voice function, as demonstrated by Verdolini-Marston and colleagues (1994), who connected vocal hygiene to improved voice function. There could be a correlation between the improved voices and the initial education of what voice hygiene is and its benefits, although that was beyond the scope of this study. There is limited evidence to date about the impact of the reduction of throat clearing or the reduction in voice usage on voice quality.

Participants had some slight improvement (everyone was "normal") in overall voice function. The monologue voice task showed most difference pre-to-post and across groups. The connected speech task, requiring greater cognitive load to create the message, may have led to less focus on clear voicing. In addition, the monologue provided more variety and usually a longer sample for scoring, which may also have affected listeners. Some variability could also be due to some participants having colds or allergies during one or both appointments, or in addition the variety of voice usage across participants due to sports.

Do Participants Rate their Voice Function Differently After Receiving Vocal Hygiene Education and Opportunities to Practice?

The Voice Handicap Index for each participant mildly improved from pre- to post- study. These results of improvement in quality of life measures, following implementation of vocal hygiene behaviors, are consistent with Behrman, Rutledge, Hembree & Sheridan (2008). Participants' knowledge of vocal hygiene was gained after the initial session, which could have affected their thinking and ratings. Since participants did complete the VHI at their initial session and again approximately three weeks later, there could have been familiarity with the VHI. Familiarity could have led participants to rate questions differently from one time to the next, so test-retest reliability could be somewhat compromised, even though the VHI tends to be a robust, repeatable measure. Participants' perception of their voice is important but was not measured directly in this study.

Limitations/Future Directions

One limitation of this study was the number and type of participants. The goal for this study was thirty participants; however, only eleven participants were recruited. In addition to a small sample size, all participants had normal voices, which does not reflect much opportunity for change in the voice. While this was intentional for a pilot study, a future study may include more participants and those who are diagnosed with a voice disorder. This would allow an opportunity to observe greater change in the voice as a result of implementing vocal hygiene behaviors and tracking those using journals.

A second limitation was that the timeframe was short. Participants completed their pre- to post-study evaluation in approximately three-to-four weeks. In designing this intervention, the time frame was chosen to capture as many participants as possible during the regular academic term and be completed prior to final exams. Further, a longer time frame may have discouraged participation due to the length and expectations of the study. All participants were enrolled as college students who were recruited in April, which was close to the end of the academic year. Future studies may wish to choose a more individualized time frame for each participant to assess how vocal hygiene outcomes are impacted by time and various disorders.

Lastly, the measures of the effect of the vocal hygiene journaling could be improved. The questions included on the daily journal felt redundant and could be reworded to gather more and different information each day. Additionally, the acoustic measures used did not seem to be sensitive to small changes in participants' voices. Although most voice function measures were identified prior to the start of the study, a few additional measures were added during analysis. It is possible that some small changes were not measured with the tools chosen for this study. While adaptations to the journal questions could be made, there are not additional acoustic measures available to improve sensitivity at this time.

Conclusion

The purpose of this study was to examine different methods of tracking patient adherence to vocal hygiene habits to understand the impact on the overall functioning of the voice. The secondary aim of the study was to evaluate the utility of a video journal as a method for monitoring home use of vocal hygiene behaviors versus using a paper journal or no journal. Journaling was effective, but the video journals didn't seem to be better than the paper journals. Vocal hygiene education seemed to be helpful for participants in order to understand the impacts of positive vocal hygiene habits. Finally, there was very limited change in participants' voice function while having a normal vocal mechanism because all participants had generally normal voice function to start.

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Appendix A

Daily journal questions

1.	Did you rest your voice today for at least 10 minutes?	Yes	No	
2.	How much throat clearing did you do today?	None	Some	Lots
3.	How much loud talking did you do today?	None	Some	Lots
4.	How do you feel your voice sounds today?			
5.	How do you think these habits are affecting you and your voice?			

Appendix B

Demographic questionnaire

1. Gender: ___________________
2. Age: _________________
3. Current voice complaints (if any): ____
4. History of voice disorders (nodules, polyps, laryngitis, etc.): ____________________________
5. Describe your typical daily voice use (consider the following):__________________________

- Singing (choir, solo)
- Yelling
- Work-related
- Telephone/videochat usage (Snapchat included)
- Public speaking
- Athletics

	No stress				Extreme stress
6. Rate your daily stress level:	1	2	3	4	5

7. Current water intake (estimated ounces): ___________
8. Current caffeine intake (estimated ounces): ______________

Appendix C

Weekly check-in questionnaire

Did you rest your voice this week for at least 10 minutes?	Yes	No	
How much throat clearing did you do this week?	None	Some	Lots
Did you feel more aware of your throat clearing?	Yes	No	
How much loud talking did you do this week?	None	Some	Lots
Did you feel more aware of your loud talking?	Yes	No	

How did you voice sound this week?

How are these habits affecting you and your voice?

Appendix D

Acoustic data rating form

Participant number:

Overall Rating	0	1	2	3	4	5
	Normal					Abnormal/severe

Comments (any unusual findings or
roughness/breathiness): ____
____

Appendix E

Strobe Rating Form

Participant ID: ________________________ Date:______________________________

Closure pattern:		Typical	Hourglass	Anterior chink	Posterior chink	Spindle	Other:
Vocal fold edge		0 Smooth/straight	1	2	3	4	5 Rough/irregular
Supraglottic Involvement		0 None	1	2	3	4	5 False vocal fold phonation
Amplitude	Left	0	1	2	3	4	5
		Normal Decrease	Slight Decrease	Moderate Decrease	Severe perceptible	Barely movement	No visible movement
	Right	0	1	2	3	4	5
Mucosal wave	Left	0	1	2	3	4	5
		Normal	Slight Decrease	Moderate Decrease	Severe Decrease	Barely perceptible	Absent
	Right	0	1	2	3	4	5

Jacket: A Poetry Collection

By:

Vincent Segovia

Mentor: Dr. Joel Pace & Dr. Bob Nowlan

Abstract

This fictional persona poetry collection depicts a Latinx veteran's struggle of transitioning into civilian life through mental, physical, and economic difficulties. Based on personal experience and influence from hispanic communities and military veterans located in Milwaukee, Wisconsin, the narration will provide a realistic take of a man's struggle with post-traumatic stress disorder, addiction, drug trade, and recovery. By applying studies within marxism, queer theory, feminist theory, and postmodernism, the collection will breakdown the start, middle, and end of an opioid addiction. The text intends on informing not only academic, but public interest in how those who suffer from opioid addiction need s rather than enforced punishment.

Writing Process

Initially, this project was to be formatted in a longform novel to depict a fictional narrative concerning a Latinx veteran returning from active duty in Afghanistan and attempts at transitioning into civilian life in Milwaukee. However, at the advisement of my mentors, I changed the format of the story from prose to a collection of poetry. More specifically, the collection uses persona poetry. Although almost every poem can be argued as a persona poem due to the freedom of expressing a narrative from separated from author as the speaker, the didactic use of a persona poem allows the speaker to take on a fictional identity; they are able to provide a plot further from the speaker through the use of fantasy. By employing this style, I was able to switch characters' identities to give personality to each while centering the story around the protagonist's struggle. used to switch between perspectives in the narrative from George, the veteran transitioning into civilian life; Melania, a sister working multiple jobs to attend college and move away from her home; John, a mixed half-brother who is a professor at a Madison Technical College; and Veva, a mother attempting to prevent the gentrification of her neighborhood.

Literature Review

I reviewed books that concentrated on drug abuse, particularly in opioid abuse from the romantic era to the 20th century. From Samuel T. Coleridge's 1797, "Kubla Khan," to William S. Burrough's *Junky,* opioid abuse has been depicted in narratives, often criticized as an unnecessary exploitation of nihilistic desires. However, in recent years such collections have been deconstructed as pleas to raise awareness as symptoms of mental illness brought on by misdiagnoses and mental illness. Most recently, Irvine Welch's *Trainspotting*, recently adapted into a film, broke down the stages of addiction from the 'ups' and 'downs' of attempts to wean one's self from the drug. With these materials, I intended on reviewing the depiction of opioid abuse to understand how the narrative structure adds to the theme while intending on preventing a repeat of the style with my own project. To further emphasize the symptoms of post-traumatic stress disorder, I read and critiqued Kurt Vonnegut's *Slaughterhouse-Five*, to apply a decentered narrative that the speaker must attempt to ground themselves in to retain agency in a civilian landscape.

Method

First, I scheduled one on one meetings with veterans and recorded their comments anonymously to prevent any form of damage to their reputation. Second, I visited Walker's Point and Clock Tower Acres to compare the similar Milwaukee neighborhoods and place an authentic setting for the narrative. Although both neighborhoods hold higher ethnic minority populations, Walker's Point had far more social programs and acceptance of separate cultures, whereas Clock Tower Acres carried less opportunities for individuals to improve their livelihood due to legislative deregulation by local government chapters within Milwaukee.

To provide authentic voices to each speaker, I used theatre movement meditation forms to create an isolationist atmosphere within my setting. In doing so, this created a sense of paranoia and outward anxiety, as well as general skepticism. By wearing my father's military fatigues, I would think over what the veteran would do in everyday life: washing the dishes, going for a walk, and receiving the news. Thus, I was able to get into character personas with first drafts using stream of thought, conversationalist poetry to depict each text as an open conversation. I used this same process when writing as a female speaker by dawning dresses with moto-jackets, applying make-up, and a wig. However, I wanted to follow traits with female empowerment by listening to post-punk bands such as Siouxsie and the Banshees, The Pretenders, and Patti Smith. After each first draft, I would move onto the next poem, later returning to the prior piece for continuous editing.

Application of Theories

Postmodernism – Although a dated term, this era of writing depicts subject matter in a non-linear narrative to break away from an overarching theme of altruistic beliefs. Rather than follow a "singular truth," or single ideology to complete and accomplish tasks, this style defects from a simplified narrative to bring forth layers of causes which lead to effects. In this case, the protagonist, George, succumbs to an opioid addiction due to a reliance on prescription drugs to relieve the physical and psychological restraints from a tour of duty. The narrator does not simply use drugs but relies on them due to inadequate transitions into civilian life. Continuing the non-linear narrative, the speaker of each poem provides little to no information as to how they entered the setting, thus depicting the psychoactive side-effects of drug use as well as assimilation into a homeland that has since become unfamiliar.

Marxism – The study of commodities applied to individuals' bodies based on their working value shows through the speaker, Melania or Mel, as an employee attempting to save up enough of a minimum wage income to attend college. However, the speaker slowly learns that they cannot afford to accomplish this as they cannot purchase items from their workplaces. The theory is also used within the setting as the neighborhood in Milwaukee faces gentrifying construction to create an interstate freeway. Since the neighboring citizens cannot argue with the city council members, they are prevented from protecting a basic income as commuters no longer have access to the storefronts.

Race Theory – Each of the characters are of Latinx heritage, however multiple speakers separate themselves from the narrative. The speaker, Veva, holds onto these traits by using terms and phrases in Spanish in the hopes of informing the readers of cross-cultural communication. However, separate speakers such as John attempt to live life as a "white-passing" individual with privilege, thus commits erasure to their identity. This separation causes a rift between the familial dynamics. The speaker, George, becomes stuck in a liminal space of wishing to embrace their identity, yet are held back due to the patriotic honor they hold to the United States as they served for their country's military service.

Feminist Theory – Although Veva holds the status as the matriarch of the family, they symbolize the passing of older waves of feminism. She wishes to remain a "homebody," however through non-linear flashbacks her history of abuse by her husband gives examples concerning how she ended up remaining within the neighborhood subject to gentrification. Melania provides a contemporary outlook on feminist movements within the 21st century; of a woman wishing to be expected of greater things than the male siblings raised socially separated by her mother, Veva. Although her voice is often repressed due to sexism within the workplace, she finds ways of creating power by blurring the lines and finding grey areas in what is considered right and wrong.

Queer Theory – This theory was applied to further exude the grey areas of gender norms, more specifically, regarding George's masculinity as a wounded soldier returning home. Trapped in a liminal space, George must ground himself by referring back to the familial bonds he built in his neighborhood before leaving for armed services with men. By blending these two environments together, George learns that his camaraderie for fellow soldiers was deeper than a familial love, but a romantic one. However, George continues to reject these ideas until once of his friends dies due to an overdose.

Examples

By concentrating on these two pieces in the collection, I wished to show the "otherness" of problems such as familial strife, ethnic identity erasure, and chronic opioid dependence. As shown in "She Follows in Hot Pursuit," the speaker, George, suffers from the loss of a lover due to their own overdose. Although he continues attempting to run from the moment of a loved one's loss, he cannot escape as he stops in a city development, similar to his, and hunted down by the guilt of accidentally killing his partner.

"Thanks-for-Givin' Arsenic" exemplifies each of the primary speakers in a single setting. Each speaker wishes to enjoy a Thanksgiving family celebration, however, also are aware of the impending argument that will arise. Instead of using the climax – depicted in a third-person narrative – I concentrated on the rising action as each speaker separates themselves from one another, thus providing a departure from the nostalgic bonds they once carried.

She Follows in Hot Pursuit

Eyes wide, bloodshot search for an eclipsed horizon
            line reflected by the night sky to direct these shaking
gripped hands across the worn leather power steering

rattling the hot tin roofed 98' Geo Tracker like a half—
            past alarm clock that refused to be hammered down
by a heavy handed "Not now."*Flipump flipump*

flipump treads below along the rolling roaming hills
            this highway was paved across, which — by law —
must be sped at least 15 miles over to reach the next

mile mark. Between the beaten tread paths, the radio
            fritz screeches its best banshee impression, piercing my
ears to meet what the brainwaves review: rosy cheeked

curves turned pale boned, riddled with needles, the punch—
            in time sheet at the Exit 13 Mobil station awaiting its turn
for rolling pins and bulldozers, and the last chunky bean

n' cheese burrito traded for the half gram snow sack keeping
            warm between both of my own that remain at the top of that list.
Aligned alongside the broken down, cracked skin county road,

sit farmland fences competing for duplicate commercial contracts
            to shelve the thickest 12oz. beef which once fertilized the rest
of the produce down beyond the frozen food aisle marked "Fresh

Organic Produce," later recalled for exposure to E. Coli.
            I can't smell the shit, nor taste shit—just make a rush for white
lights peeking over the last hill. Yellow and red lights advertise

a slab of grease between 2 buns and veggie alternatives embroiled
    on the same stained, grill timers. Adjunct stand towering window
patterned, tattered buildings competing for the lowest rate stay.

Concrete roads held by crumbling support beams web above to mark
    which way is right. With another lick of the snowsack, I hit the gas
for the exit onto the interstate, yet no sign peaks in sight. Only a large

green hand of grass blocks my path at 60 miles an hour. The Geo
    nearly topples into the embankment of weeds, gravel flies, and I come
to terms that my death certificate will be marked "Death by Failure,"

before my arms swing the wheel to a hard left onto the interstate exit.
    A Jewish man sings of his friend named Jesus as my eyes lock with a crying
silhouette ripping away tufts of grass from an arrow sign pointing left only.

Thanks-for-Givin' Arsenic

Georgie's Last Methadone

"Smoke 'em if ya' got 'em"
    drill sgt's words still ring
from then   I didn't smoke

not then   savages draped
    in RPG's we supplied
to see the red in white 'n blue

    now the smouldering ashes
burn between fingers   where
    The spoon suppose'

to sit   where the IKEA fork

        Will be   smokes'll
be the only menu's entree

for   tonight's dinner
        for tomorrow
that next clinical hit

Melanie's 5 More Minutes

"Where ya' goin' shorty"
        called from aisle
3 littered with 'em boys

those kiddies I sweep
    with a smile,
that wink they know

not to bite at, because
    mine licks lips
just to shatter teeth set

on the pavement I
    just finished
sweeping and lord

knows Boss'man
    can't lose
another gas clerk

just before dinner
    'fore John,
Georgie, and mom

John's I-90 Exit from St. Paul

Already my knuckles
   crack, bleed
Milwaukee's suppose'

to be warmer than St.
   Paul. I'll kill
channel 5's weather

man and Georgie's
   gotta' hold
the last word as he

but this is not high
   school time
and I'm not shoved

in that ivory tower
   locker Mel
joked about last year

Homemade Stuffing

I know this will
go well like it has every
   last time before

   John was only
joking that lil' cynic
   he's spending

   too much time
up there anyways. He

lost his touch

Mel's gotta'
settle down, at least
soon. try not

to bring that up
Veva and my Georgie.
he hasn't been

Home not since
making his landing here
some place safe

Dissemination

By the spring, there will be a total of 30 to 40 poems that will switch from separate perspectives to depict the beginning, middle, and end of the addiction. Before submitting this collection to publishing agencies, I intend on sending individual pieces from the collection to Barstow and Grand, Volume One, and the Chippewa Valley Writers' Guild for additional recommendations. As a full collection, I plan to submit them to Dorrance Publishing, Balboa Press, Fulton Books, and fifteen other publishing agencies outside of the Midwest of the United States.

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Exploring Prosociality in Osage Culture

By:

Edward Sobottka

Mentor: Dr. Heather Ann Moody

I am going to start this off with a personal story. I do this in order to contextualize this research and my place in it. I am Osage, but I was raised apart from my tribe both physically and culturally. Physically apart in that I grew up multiple states away from where the Osage now reside in Oklahoma. Culturally apart in that I was raised in the dominant U.S. culture and learned little about Osage values. I grew up watching my parents help others when they could, and helping others became a part of my identity without much thought as to why. Reflecting on my own motives for helping and my desire to better understand my cultural heritage fostered my interest in researching Osage helping motives.

In doing initial research for this project I learned that scholarly research on Indigenous people has almost entirely focused on the negative aspects of Indigenous life. The Osage Nation's website contains a list of research done in their community and although it is not exhaustive, research topics favor special ceremonies, rites, and art whilst leaving out helping. Many disciplines, such as Economics and American Indian Studies have researched helping, but Psychology has dominated the topic. Some of the most commonly used terms in the field are 'helping,' 'empathy,' and 'prosocial behavior' and searching for them in the PyscINFO database yields a combined total of 54,536 results of peer reviewed research. However, simply adding 'Indigenous' to each of those search terms returned 177 peer reviewed results. Surprisingly (given the prevalence in colloquial usage) searching 'Native American' instead of 'Indigenous' yields only 99 peer reviewed results. Note that these totals excluded non-peer reviewed results and included potentially less relevant peer reviewed research. This will certainly change the total number of results but is not likely to change the pattern of them. To gain the most complete understanding of how and why people help others, more work needs to be done to include non-dominant perspectives. To that end, my primary goal for this research was to learn about the Osage perspective on helping, perhaps painting a more positive picture of Indigenous people

In summation: the purpose of this research was to help fill the gap in knowledge on helping by examining how Osage people define helping and what motivates Osage helping behaviors. I speak to how this was done in the Research Methods section but an understanding of the definition of helping is necessary to begin. Following that, motives for helping will be discussed. Since how and why people help others varies based on culture, a review of ethnic group differences will be presented followed by a look at non-Osage, Indigenous perspectives on helping. Because "help" is a vague term encompassing myriad behaviors we should look more closely at what help can mean.

Defining Help

In scholarly research helping is often referred to as prosocial behavior. As a more formal term, prosocial behavior is more descriptive of help's function. Prosocial behavior functions as a method to make it easier for one to do something. It need not come from a human. For example, properly trained dogs can make it easier for the visually impaired to move safely. A person can help themselves or others, either intentionally or accidentally. In addition, help can be dependency-oriented or autonomy-oriented. Although they did not coin these terms, Arie Nadler and Samer Halabi (2006) gave a good definition of them in writing that dependency-oriented help provides a full solution to the problem whereas autonomy-oriented help provides a partial solution.

Think of one mother (Susan) baking cookies *for* her daughter and another mother (Donell) baking cookies *with* her daughter. Considering the definitions above we see that Susan is providing dependency-oriented help whereas Donell is providing autonomy-oriented help. This example introduces the concept that some motives can influence *how* we help. I elaborate on this in the Intergroup Helping section, but prudence dictates that we first examine motives for *why* we help.

Interpersonal Motives

As mentioned previously, helping has been researched by many, covering disciplines from Business to Psychology. With many theories on the motives that drive helping behaviors, it can be hard to discern which are ultimately important; however, two concepts based on theories of moral development stand out. Sociologist Mark Ottoni-Wilhelm and Economist René Bekkers (2010) identify these concepts as dispositional empathic concern (DEC) and a principle of care (PoC). Under DEC people help because they tend to feel some reactive attitude, such as sympathy or concern, as a response to others' needs. This is contrasted with the PoC in which people help because they possess an internalized value to help those in need.

One should note that neither concept is inherently better than the other; they interact with each other but typically have different aims. Help brought on by DEC is associated with spontaneous, short-term actions, and is typically aimed at people concretely known to the helper. For example, allowing a stranger with fewer groceries than you go ahead of you in the checkout line (Wilhelm & Bekkers, 2010). On the other hand, help brought on by a PoC is associated with planned, long-term actions, and is typically aimed at people the helper abstractly knows. For example, donating money to a charity that helps people in need (Bekkers & Wilhelm, 2016).

Participants in these works were nationally representative samples of American (Wilhelm & Bekkers, 2010) and American and Dutch peoples (Bekkers & Wilhelm, 2016). Typically, nationally representative samples group all Indigenous nations together, making it impossible to differentiate between individual nations. This is significant insofar as the current research focuses on the Osage Nation so that specific differences between Indigenous nations may be understood. Moreover, Indigenous nations typically emphasize sustainability (whereas Western worldviews typically emphasize growth), and successfully maintaining a sustainable society requires more planned, long-term behaviors. I elaborate more on Indigenous worldviews in the Indigenous Perspectives section, but the relevance here suggests that because the Osage are Indigenous, they are likely to hold similar views on sustainability. Thus, Osage helping will likely be motived by a PoC more than DEC. However, because there may be some overlap in these concepts (i.e. a person may help because they believe they should, and they feel sympathy for someone who needs help) Osage helping will likely contain both motives. To distinguish between DEC and a PoC motives in Osage helping behaviors, I follow Bekkers and Wilhelm in defining DEC as "the tendency to experience concerned, sympathetic, or compassionate reactive outcomes in response to the needs of others" (Wilhelm & Bekkers, 2010) and a PoC as "the moral position that one should help those in need" (Wilhelm & Bekkers, 2010). I believe DEC and PoC are motives for why people help but other factors may influence how and when people help.

Intergroup Motives

If one reflects on times when they have decided to help (or not) they may find motivating factors are often more complex than the simple DEC/PoC dynamic. Constructs in Social Identity Theory (SIT), like status relations between groups, individual levels of identification with a group, and the type of help, can complicate care-helping and empathy-helping relationships. In their 2006 study, Psychologists Nadler and Halabi conducted four experiments to assist in explaining how each construct adds to the complex relationship between willingness to help and actual helping behavior.

The first experiment focused on perceptions of the stability of status relations. The results showed that, in experimentally created conditions, when status was perceived as stable, high-status outgroup help had no influence on the recipient. However, when status was perceived as unstable, receiving help from high-status outgroups made the recipients feel worse. The second and third experiments focused on the relationship between helping and group identification in real groups of Israeli Arabs and Israeli Jews. Group identification refers to the extent that one identifies as a member of a given group. The results from the second experiment replicated the findings of the first experiment in real groups of people. The results from the third experiment suggested that high identifiers in a low-status group (Israeli Arabs in this case) felt worse after receiving help from high-status outgroup members (i.e. Israeli Jews). Experiment four focused on low-status group members' willingness to seek autonomy- and dependency-oriented help from high-status outgroup members and continued to check status stability and group identification. Results showed that no high identifying participant sought dependency-oriented help from high-status outgroup members in unstable status conditions.

Taken together, these data clearly illustrate the complexity of motives behind prosocial behavior. As power dynamics, group identification, and type of help are likely to affect the strength of one's empathy, change who needs help, and how the help occurs, separate from the development of empathy and a desire to help, it is my view that DEC and PoC are more foundational than the constructs of SIT. While power dynamics, group identification, and type of help are relevant motives and worth noting, a limitation of the current research will be that it will focus on DEC and a PoC. Neither interpersonal motives nor intergroup motives adequately addresses ethnic group differences but there is reason to believe that ethnicity matters when it comes to helping.

Ethnic Group Differences

A complete understanding of the motives behind helping others necessitates examining potential ethnic group differences. Business ethics research done by Singh and Winkel (2012) focused on antecedents of helping behavior in the workplace and how to predict interpersonal helping behaviors (IHBs) focusing on ethnic group differences. They rightly claimed, "it is important for researchers and practitioners to understand the impact of racial differences on workplace phenomena" (Singh & Winkel, 2012, p. 468). Although they gave participants six categories (Caucasian, African American, Native American, Hispanic/Latino, Asian/Asian American, and other) they stated that, "for the purpose of analyses, the racial affiliations were further categorized into two groups: majority versus minority" (Singh & Winkel, 2012, p. 472). Regardless of whether this was necessary to obtain appropriate statistical power, by collapsing minorities together the researchers appeared less sensitive to ethnic group differences than they claimed.

Moreover, by dichotomizing their analyses they continued a tradition that otherizes non-White peoples. To me, their claim that "the results reiterate the fact that when dealing with diversity, 'one size does not fit all' (Mckay et al. 2007)" (Singh & Winkel, 2012, p. 475) becomes a poignant reminder of a limitation in their research design in addition to the advice for researchers and practitioners they intended it to be. I take this limitation to inform how the current research is presented. For example, couching the relationship between interpersonal and intergroup motives as interconnected instead of adversarial.

As instructive as this limitation is, their results are also informative. The results suggested that mutual respect and psychological safety (a person's ability to be themselves without fear of negative consequences) are determinants of IHBs and that this relationship is stronger for ethnic minorities (Singh & Winkel, 2012). These data indicate that by respecting the Osage worldview, working and learning with the community, and encouraging openness and honesty without judgement Osage people may be more likely to give honest responses. I believe this will work towards counteracting the social desirability bias that is inherent in the self-report design of the current research. Singh and Winkel (2012) grouped all ethnic minorities in one group but as I mentioned previously, one goal of this research is to distinguish Indigenous nations. Thus, it is now appropriate to review non-Osage Indigenous perspectives.

Indigenous Perspectives

Oren Lyons of the Onondaga nation said they were told that if all human beings got along with each other and supported the laws of the natural world "life is endless" (as cited in Nelson, 2008, p. 24). Cree activist Priscilla Settee indicated that Northern Athabascan worldviews focus on, "good family relations.…sharing, caring, [and] village cooperation" (as cited in Nelson, 2008, p. 46). Further, Jeannette Armstrong stated, "we can't call ourselves Okanagan if we can't provide for the weak and the sick and the hungry and the old and the people who do not have skills" (as cited in Nelson, 2008, p. 70). John Mohawk, a member of the Turtle Clan of Seneca Nation of New York, took a harder stance, "The Peacemaker said that the problem that humanity faces – and all humanity faces this problem – is that the absence of peace will lead to the end of human life on this planet" (as cited in Nelson, 2008, p. 57). Anthropologist Megan Biesele has said that a lesson the Western world can take from the Ju/'hoan San people is that we all live together, and we have to depend on each other (as cited in Nelson, 2008, p.75).

I present these views together to illustrate that many Indigenous societies emphasize sustainability which demands we help each other. In helping the weak, sick, hungry, old, and unskilled, Armstrong seems to imply an internalized value to help those in need (i.e. a principle of care) is essential to being Okanagan. Some might argue that helping is not necessary for the peace John Mohawk mentions, and we could simply set up borders and leave each other alone, but for that reason I included the lesson from the Ju/'hoan San people. Because we live together on this planet, we will inevitably need help from others and so we should help others in return. The Indigenous views in this section come from American (Onandaga and Seneca), Canadian (Okanagan), and African (Ju/'hoan San) nations and are relatively similar but not entirely the same. For this reason, it seems reasonable to conclude that the Osage worldview and Osage helping motives will be similar to other Indigenous peoples.

Research Methods

For the current research, data was gathered in-person via paper survey and focus groups. In addition to providing demographic data, the survey contained an open-ended question asking how the participant defined helping. The survey also included a 10-item list of behaviors that have been associated with helping in previous research (Bekkers & Willhelm, 2010; 2016). Participants were instructed to mark all the behaviors they believed were helping behaviors. This list was included to provide additional data about Osage definitions of helping.

In his 2008 book *Research is Ceremony: Indigenous Research Methods*, Dr. Shawn Wilson (Opaskwayak Cree) outlined the importance of stories in Indigenous cultures and provided a model for ethical Indigenous research. Thus, the current research involved audio recording contemporary and traditional Osage stories during focus group discussions to address participants motives for helping. Contemporary stories were the stories about the lived experiences of participants, whereas traditional stories were stories passed down from previous generations. Following Wilson's guidelines, participant recruitment followed Osage cultural values. This meant that I started by asking my family if they wanted to participant (they declined) then moved to people they introduced me to. Using this snowball style convenience sample, I recruited 8 participants aged 19-80 (M = 45.5) for the survey and 6 of those volunteered to participant in the focus group discussions. The focus group participants were split into two groups based on their availability and, due to lack of time there was only one session per group.

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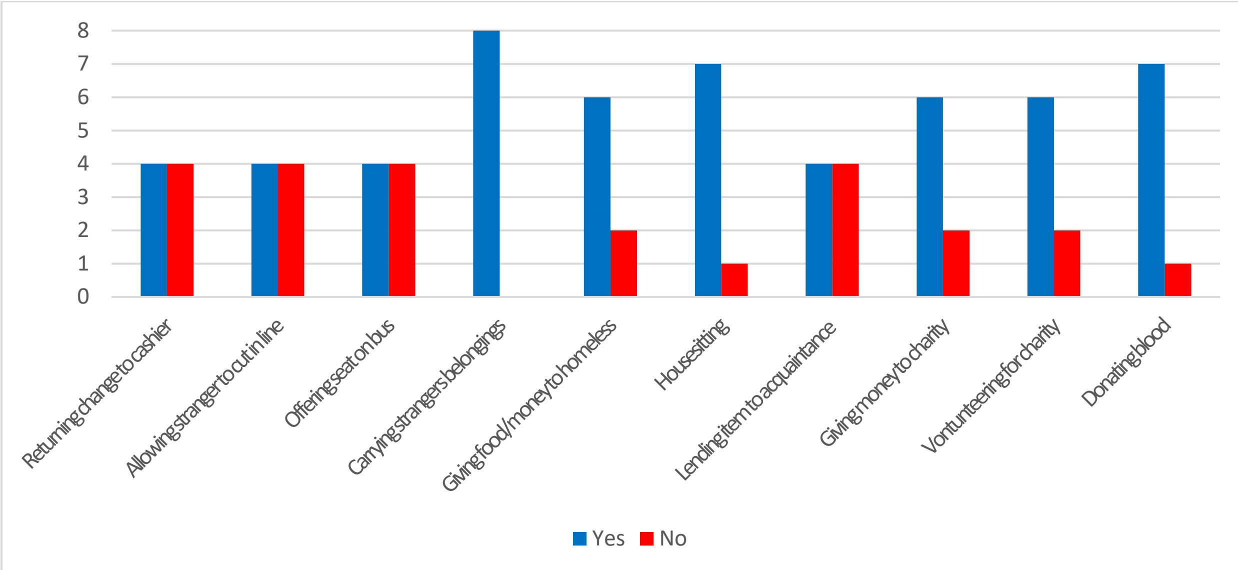

Figure 1. Frequency of behaviors considering helping.

Results

Based on the open-ended question, participants most commonly defined helping as assistance given during a time of need. Participants also indicated that this help could be with something big or small, and could take a variety of forms, including physical (e.g., helping a friend move) and emotional (e.g., offering encouragement). Figure 1 shows the frequency of behaviors marked on the 10 item checklist. Although there was some variation, each behavior was considered helping. Interestingly, only carrying a stranger's belongings (e.g., helping take groceries to their car) was considered helping by every participant

Participants had some difficulty in sharing both contemporary and traditional stories related to helping which I will expand on when discussing the limitations of the current research. Because there were only a few stories shared, contemporary and traditional stories were grouped together and analyzed for common themes. Through this analysis one minor and two major themes were identified.

Major Theme: Upbringing

By far the most common theme motivating participant helping was their upbringing. Every participant talked about being raised to help others, and some indicated that they were not supposed to think too much about it. Amidst sharing when she has helped others, Alaina (group 1) said, "*that's how my family taught us. Just help your family whenever they need help."* Christopher (group 1), one of the younger participants, stated, "*But for me…growing up it was just like, if you can help somebody, do it, ya know? Don't sit and wait. Don't think about it. If you're right there and you can lend out your hand, do it.*" An older participant, Vann (group 2) added to this saying, "*we're all…always taught to, I think, to help other people. That's kinda the way I was brought up. I just try to be that kind of person*." John (group 1), mentioned that "*when you grow up in an Indian house you help each other*" illustrating that helping is an integral part of Osage life.

Major Theme: Survival

The second major theme revealed in group discussion related to survival. Speaking primarily about previous generations of Osage, Janis (group 2) said, "*I think, historically, that Osages have always been helping people because…they had to, to help their families survive but…it didn't stop there. It went on to their village, the people they lived with, the people they depended on because they were depending on each other for everything, for food, for clothing, for survival."* Janis tied these past needs with current motives stating, "*I think we all know how important that is to help each other."* John added to this theme saying, "*That's how you're able to survive as a free people for so long, is you have to depend on each other and help out."*

Minor Theme: Humility

I felt it important to include humility as a minor theme for two reasons. First, because it was a novel motive for helping that I had not expected. I classify it as minor in comparison to the other themes because although multiple participants discussed the significance of humility as an Osage cultural value only one explicitly mentioned it as a reason for helping others. Alaina stated that because of the value on humility Osage people are taught to put others first and "that's why we help people." Second, as was mentioned earlier, humility has a significant role in Osage society. Thus, while it was not explored in this study further research could provide a better understanding of the relationship between humility and helping behaviors.

Discussion

As stated earlier, the primary goal for this research was to help fill the gap in prosocial research by exploring Osage helping behaviors and motives. The research that has been done with other populations shows that 'help' is a broad term, encompassing many definitions, behaviors, and motives. To my knowledge, the current research is the first of its kind to apply this knowledge to the Osage people. Through the survey, participants revealed that how they define helping and what behaviors they consider helping is as varied as other populations. Furthermore, that depending on each other for survival was a major theme of the focus group discussions suggests that Osage motives for helping are similar to other Indigenous peoples. However, it is important to note that no comparative analysis was performed so proper comparative claims are limited.

The current research was limited in a multitude of other ways. First, the Osage Nation website indicates that the Osage population is close to 20,000 members so the views of the 8 participants in the current research may not represent the true diversity of prosociality in Osage society. Relatedly, the sampling method, while appropriate for the population, still comes with certain limitations. As a type of convenience sample, being introduced to potential participants through my family limits the generalizability of the conclusions made. Family members and their closest friends often hold similar views, so it is possible that definitions and motives discussed above do not generalize to other Osage people.

Second, participants spontaneously reported that they were taught to simply help when they can and not think much about it. This presents a number of limitations. When introducing this research to and discussing it with the participants many professed difficulties defining help. If given more time to think about the topic it is possible that the same participants would report different or more refined definitions for 'help.' It is also possible that this difficulty came from how the topic was discussed. Although I utilized an Indigenous research methodology, the current study and my perspective is largely rooted in a Eurocentric worldview. Many Indigenous perspectives are centered around harmony and balance (Nelson, 2008) so perhaps the more traditionally raised participants in this study would have been more receptive and comfortable talking about prosocial behavior in these terms.

Based on the history of dominant American society's disinterest in Native perspectives (Wilson, 1988) it is most appropriate to use an Indigenous research methodology (Smith, 1999; Wilson, 2008) for research with Indigenous populations. A foundational element of this methodology is learning <u>with</u> the participants not <u>from</u> them, often achieved by involving participants in more of the research, effectively making them co-researchers. Because I was a student in Wisconsin, with few connections to Osage people when creating the current research, I was primarily designing it by myself with some direction from the Traditional Cultural Advisors Committee (TCA), a group of Osage elders from the three districts who advise on research and other matters conducted in The Osage Nation. Future works would do well involve the TCA and others in the community in more of the process. Working with them to develop the proper questions to ask and how to best phrase them would better situate this research in an Osage context, making it easier for participants to discuss the topic. It would also give them more familiarity with the concepts which was a significant limitation for the current research. Involving the TCA and participants from the current research in more of the design process aligns with another major component of Indigenous research.

Relationality is a key aspect of Indigenous research because connections between people, animals, and nature are key to Indigenous worldviews (Smith, 1999; Wilson, 2008). By involving Osage people in more of the research I hope to strengthen the relationships I have already developed and start new relationships within the community. This has two major benefits. First, by building on existing relationships we learn to better trust each other. Through personal experience I know that individual relationships are not the only way relationality is understood by Osage people. During my time in Oklahoma I met many people, and one of the first questions I was asked was about my family. Knowing familial relations is important because it provides a basis for an understanding and trusting relationship. With this trust comes a better understanding of how this research and the information learned by it will be used. Thus, participants could feel safe in knowing that what they share is not to be viewed as boastful but educational. This works to correct the limitation that humility may have had on the current research. Second, because I already have established some relationships with the Osage community, building on them increases the potential pool of participants. As more people participate in this research it becomes more representative of the whole of Osage culture.

Understanding Osage history is important because it provides a basis for understanding the significance of prosocial behavior in Osage culture but focusing solely on history limits the ability to know how this value has changed over time and what motives undergird prosocial behavior in contemporary Osage society. Furthermore, Indigenous people on the land now known as America fight a narrative that situates them in the past, creating a sense in the dominant society that they are no longer living. Because Indigenous peoples still very much exist today it is important to include in research a narrative of contemporary Indigenous peoples. Thus, to best understand the motives underlying Osage prosocial behavior, a deep understanding of past and contemporary Osage society is necessary.

Third, the value of humility in Osage culture limited the stories shared during focus group discussions. As Chris stated, "*…any instances where I've really helped anybody…I've never really talked about it. Cause I always felt like I didn't want to boast about it.*" Janis expanded on the interaction between helping and humility. She described how having a reputation for helping others can increase one's social status but personally boasting about how you help others is frowned upon. Analyzing these stories was the intended method for understanding Osage helping motives but the few stories shared limited the analysis that could be done. Future research would do well to consider the role of humility to inform the methodology used and phrasing of questions. Finally, it should be noted that the current research was an exploratory study and as such opens up many avenues for future research.

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The Power of a Name

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Abstract

This research project, The Power of a Name, is focused on the cultural dissonance felt by people of color through the anglicization of their names in American institutions. It stems from the observation that there has been a history of American colonization of marginalized people's identities for centuries that has enforced the assimilation and acculturalization of these groups. This project aims to look at this aspect of colonization and research the history of renaming oppressed groups in the United States to show the progress that led to the more contemporary anglicization of immigrant children's names in public schools. This resulted in an exploration of how historical and contemporary renaming has influenced national identity, and self-perception in multilingual school- and college- aged individuals through the study of Critical Race and Postcolonial theorists and interviews with UWEC multicultural students.

I. Introduction

The colonization of a body has always been a product of the colonization of a land and this results in the forceful assimilation and acculturation of a people. Many of our institutions have been set up to enforce American ideologies and perpetrate them upon marginalized people and one can observe how the structures created during colonialism still exist in the infrastructures of today's society. This project primarily focuses on how the changing of a marginalized person's name to an anglicized version or pronunciation is part of this larger system of colonization. Historically, this form of colonization has claimed the identities of marginalized people by stripping them of the cultural significance of their names as well as imposing new ideals of identity. By understanding how the naming of enslaved people or the renaming of American Indians has set up systems of disidentification, we may begin to tie these structures to critical race theory as well as how these systems affect us today. My project aims to show how the anglicization of one's name is a product of colonization that has historically destabilized marginalized people's identities and this destabilization continues to be perpetuated by institutions today.

II. Theorists and Terms

Jose Esteban Munoz introduces the idea for "performing disidentifications" as a way to subvert these colonial institutions (Munoz 25). He argues that much of identity is performative and from this comes the idea of disidentifying which he describes as "apolitical sidestepping, trying to avoid the trap of assimilating or adhering to different separatist or nationalist ideologies" (18). A person's identity is individualistic and the idea of disidentification is a mode of survival to escape the systems that "reduce identities to lowest-common-denominator terms" and that the performance of disidentification exists "at precisely the point where the discourses of essentialism and constructivism short-circuit" (6). Munoz points to the performance of these intersections as a way to counter the colonial pressure to assimilate by affirming one's existence in multiple cultures.

Munoz continues by introducing terms like "differential consciousness," and "hybrid lives" which he uses to identify the multiplicity of selves in a performance (6, 31). The first of the two terms identifies the "fractured and split" identities that many cultural scholars note, and which one can see in the "racial confusion" of Native American children in boarding schools (31). This concept is one which Chela Sandoval chose to describe the Chicana's experience in a world that where they are multiply oppressed as a woman of color across multiple cultures. It is the idea of moving fluidly through cultures by knowing when to emphasize and de-emphasize aspects of one's identity to fit within the constraints of one's current surroundings. The idea of "hybridity" is Munoz's response to Sandoval as it offers a more holistic approach to the individual in that it does not ask for one performance to be heightened above others but instead to perform as one and capture the multiplicity of identities.

Much of Munoz's antiassimilationist rhetoric is drawn from work by Gloria Anzaldua who further explores the fluidity, ambiguity and ambivalence of cultural identity. Unlike Sandoval, Anzaldua portrays "differential consciousness" or as she calls it "dual identity" as a consequence of "not acculturating" and its presence is a force that people of color must resist as it results in internalized conflict between the selves (Anzaldua 85). There is "the stress of cultural ambiguity" that goes with ethnocentric ideology, that one must choose which identity to perform to cater to the majority's preference in any given situation and that there is a sort of sacrifice that people of color must make to be fully accepted. She goes on to argue that there needs to be a "tolerance for ambiguity" and people must become comfortable with their "pluralistic mode" of existence (101). Anzaldua and Munoz both reject this idea of sacrificing the self, although they understand its importance in survival, they argue that to dismantle the systems within our institutions, one must be allowed to express their multiplicities of identity.

Munoz and Anzaldua do differ slightly, in that Munoz believes language is a part of cultural performance while Anzaldua affirms that "ethnic identity is twin to linguistic identity" and that you cannot separate a language or culture from an individual (Anzaldua 81). That is, a language is more than performance, it is the foundation on which one builds their identity. Anzaldua's Chicano Spanish is a language that defines her, and her multiple identities and she positions it as a cultural anchor that ties her to what it means to be Chicana. She acknowledges the violence of stripping one of their language and how the creations of patios and pidgin languages creates "a homeland" for those who have migrated (77). Anzaldua sees language as an uncontrollable aspect of one's identity, and not as something that should be performed or sacrificed regardless of the circumstance, stating, "until I am free to write bilingually and to switch codes without having always to translate… and as long as I have to accommodate the English speakers rather than having them accommodate me my tongue will be illegitimate" (81). In this statement, Anzaldua introduces code-switching and its role in identity and how it represents the multiplicity of self in the way one speaks. This point is especially important when one considers the historical implications of stripping a culture by banning the use of a people's language, and with that usually comes the anglicization of their names. While identity theory did not reach mainstream academia until the mid-twentieth century, America's history built the frameworks that would destabilize minority group identities from it's very foundation.

III. Historical Context

One event that began the creation of these frameworks is the naming of African slaves in American and British colonies. Research revealed the dehumanizing forms of naming that slaveholders in Jamaica used to strip slaves of not only their culture but also their individuality. Common practices included "clustered naming" wherein groups of people who were similar in value, age, or ability were given names from the same "pool" such as days of the week, months, or characters in a play or novel (Williamson 120). Occasionally, these "classical" names, that is those derived from plays and novels, were used "as a kind of in-group joke" where it was "common practice with the colonists to give ridiculous names… to their slaves" (122). This exertion of power through perceived intellectual superiority further heightened the dehumanization of enslaved people and turned their own name into a point of mockery.

Additionally, there were forms of naming at the time that were more subtle in their approach to alienating enslaved people from their culture. In some cases, slaves were given "hypocoristic (pet or diminutive) versions of English names," a practice which infantilizes and condescends their identity (Williamson 117). This practice perpetuates the idea that people of color are lesser versions of their colonizers and by giving them these diminutive forms of English names it forces their identity within this subservient role. Furthermore, renaming of slaves became common practice and was even seen as "routine" and yet was still considered "an act of appropriation analogous with branding" (119). The culture of indifference to this act further ingrained it into the society and after some time "both slaves and owners came to see African names primarily as signifying slave status rather than a source of pride" effectively removing enslaved people's ties to their cultural identity as well as the ability to even claim a cultural identity at all (118). Slavery as an institution shaped the way enslaved people perceived themselves and the process of naming stripped them of their culture and their individuality.

Another institution that influenced the acculturation of a marginalized group is the creation of Native American boarding schools in the United States and Canada. These schools, first founded by Colonel Richard Pratt who believed the way to help the Native people was to "kill the Indian, leave the man" and "the intent of the western educational system was to purposefully eliminate the cultural identity of American Indian people" by using several methods to systematically strip students of their culture, including giving them new, English names (Smith 57). These methods of acculturation created "racial confusion for Indian students" and greatly influenced how they saw themselves as well as those in their community. Additionally, there was a "constant conflict between what white educators taught and what was taught in the home culture" creating a rift in the student's perception of themselves (59). Students were told that "everything Indian was viewed as negative" and this created major rifts in their connections back home (59). Tribal communities were greatly affected as the boarding schools took away the children's ties to their family, language, and identity.

One of the main tactics that the boarding school system used to cut Native children's cultural ties was to strip them of their "cultural anchors" which included banning their language and changing their names (61). American educators understood that "Native culture resides in the Native language" and the banning of the students' languages in the schools resulted in the loss of many indigenous languages and with that, the loss of their cultures (61). This greatly affected the tribal communities as many children returned from these schools having practically forgotten their native language. Additionally, the enforcement of an English only school led to the forceful renaming of the students "because their Indian names were difficult for teachers to pronounce" (63). Not only was this a culturally insensitive practice since "in many tribes, the names given to individuals are viewed as sacred and given by the creator" but it was also dehumanizing in the way it was conducted. In several cases "school officials would have students stand in a line and give them names according to the alphabet," showing the children that a name given to them at random is more important than the name given to them at birth simply because it is English rather than in their native language (63). This treatment resulted in decades of inherited trauma and continued disregard for Native culture and language.

Over time, name changes still happened for marginalized groups in America but the way they happened changed shape. Whereas before, name changes were forced upon people, the industrial boom in the early 20th century gave 'incentive' to change to more English/Christian names. These name changes among Jewish and Italian immigrants during this time "need not imply assimilation" but it did show a sort of self-monitoring that developed in the immigrant consciousness over time (Watkins & London 172). During this time, it might have been expected that people would change their names "in order to avoid economic discrimination or to gain economic advantage" but if that were the only influence, one might expect records to show more Italian and Jewish men changing their names when, in fact, the change came more rapidly for women (173). Language change usually happens more quickly among women, but tradition also played a large role in slowing name changes for men, family names were usually passed down from the father's side, whereas women's names became a form of "verbal jewelry," that is, they were more likely to be chosen for aesthetic preference (173). This fact reveals what names were seen as 'beautiful' for those who came to America and how perceptions of names changed for immigrants as they were socialized into American culture. Traditional Jewish and Italian names were no longer seen as 'beautiful' names for second-generation immigrant women, but this did not mean that immigrant families were quick to abandon traditional names.

It was not uncommon in the early 20th century for immigrant children to have two names, their traditional name and their American name. In an interview with a first-generation Italian immigrant (born in 1905), she recalls calling her sister Lucy at home but knowing that it was a stand in for the traditional Italian name, Lucia (Watkins & London 176). This was one way that immigrants subtly refused to fully assimilate and protect one aspect of their culture, even if it was not the name they were called at home, it was the name that was put down on paper. This also lines up with the idea that immigrants' names were changed upon arrival to Ellis Island which is not entirely the case. Many new immigrants may have chosen to change their names upon arriving, a practice that is still available through the naturalization process today, giving their chosen anglicized name but still going by their given names at home while others did the opposite. Having two names as the norm among these groups also created a sense of a new shared identity, one that was not from their original home but also one that was different from their new surroundings, it helped to create a separate immigrant identity. This use of alternate names would become a prominent tool for many immigrants and lines up with Munoz's ideas of disidentification as well as Anzaldua's focus on the importance of code-switching for immigrant communities.

IV. Contemporary Context

Code-switching is the mingling of two or more languages in a multilingual person's speech and is often seen as "an in-group device, typically restricted to those who share the same expectations and rules for interpretation" (Benson 310). Because of its use among marginalized groups, code-switching is often stigmatized and discouraged, especially within academic institutions (311). This concept is important to note in this study because as one's name comes from their native language, the correct pronunciation of it in an English-speaking context may be seen as code-switching. Anzaldua herself recalls an instance wherein she was reprimanded by a white teacher when she attempted to tell her how to pronounce her name (Anzaldua 75). The stigma surrounding code-switching may offer a possible reason for why the anglicization of names continues today and this stigma against code-switching further perpetuates the acculturation of marginalized groups.

These restrictions on code-switching and multilingualism have long existed within academic institutions, as seen in the United States' boarding schools, and continue to be propagated through schools and pedagogy, resulting in children growing with internalized negative perceptions of their identity. The mispronunciation of marginalized student's names has led to "anxiety and even resentment" toward their identity and culture (Kholi 442). Many learn to anglicize their own name early in their academic careers and this change further destabilizes the child from their cultural identity. This is often the result of students internalizing the microaggressions that they face surrounding their name and language and is offered as an easier alternative for their white peers. Students are taught that they must forfeit this aspect of their identity to fit within the larger populous and it commonly results in Sandoval's 'differential consciousness' rather than Munoz and Anzaldua's idea of an ambiguous, fluid, and unified self. Although institutions are making the push to be more inclusive to students of diverse backgrounds with pedagogy moving to support multilingual learning and code-switching in academic settings, the student's own name is regularly forgotten.

After interviewing college aged students at the University of Wisconsin-Eau Claire, a predominately white institution, one sees how the history of assimilation has impacted marginalized identities. Interviewing these students meant speaking to individuals who were raised within the Western education system and their responses reflect how this system impacted their view on their names as well as the way the expressed their cultures. In total, I interviewed five students from four different ethnicities: Hmong, Native American (HoChunk), Mexican, and Nigerian (Yoruba and Hausa/Funali). Of these five, two were second generation immigrants and two were first generation. The interviews lasted between 45 minutes and an hour, during which I asked a series of questions, at first centering on the individual's name, then gradually expanding the scope to address their perception of their own identity within the context of their surroundings. From the interviews, major connections were drawn between historical as well as theoretical contexts and, while the group was fairly diverse, many of their responses reflected similar ideas surrounding identity.

One concept that emerged from these interviews was the idea of names as "a rhetorical device" as one participant put it, similar to Jose Munoz's disidentification and Chela Sandoval's differential consciousness, students chose when to emphasize and de-emphasize parts of their identity through their names as a mode of survival. Of the participants, three had anglicized nicknames that they used to substitute their given name. Two of these cases were of students who had two names, an American name and a traditional one, the other was a simplified version of their given name. All three of these students had a variety of criteria that dictated which name they used depending on the context, who they were speaking to, and what outcome they were looking to achieve. The single criterion that had the most influence on how they introduced themselves was of course whether or not they were talking to someone from their in-group. All students noted that even if they were speaking to person of color, they would still default to their American name unless they person was from their specific ethnic group. When asked if it was out of concern for their safety, one student responded with "I can't change the way I look, I can just change the way my name sounds... It's not safety, it's just trying to be approachable" while another said "no, the only thing I'm saving is myself from hearing [my name] butchered." Both these comments make the same point though, although through different perspectives, these students did feel a pressure to make their names easier to pronounce for an English speaker.

All students were asked if they tried to make their names "easier to pronounce" --the question was intentionally left vague about what this phrase meant but all took it to mean more anglicized—and all students could recall an instance where they did do so. One commented "I just wish my name was prettier, a name that was easy to say like [my nickname]" revealing how their understanding of what constituted a 'pretty' name was one that was easy to pronounce in English. These themes of approachability and acceptance continued to come up throughout the interviews, showing how participants formed their perceptions of their culture and language within the colonial constructs of the dominant culture. When asked what might happen if they introduced themselves by their traditional name, one student, who was born in the U.S., laughed before saying "I used to but then I'd get these looks… like I was from a different planet" and this was enough to influence how one saw their name as foreign or different even if they knew nothing besides American life.

Of the five, two went by their given names, although both noted that their names were not too difficult to pronounce in English which made it easier for them. This did not leave them exempt from the anxieties that the other participants expressed, one noting how her name, although easy to pronounce, is very traditionally Muslim which concerned her family when she chose to study in the U.S. The other commented on how this created a different kind of anxiety, that he might not come off as "Mexican *enough*" because his name appeared Americanized saying, "unless you asked or you knew already, you wouldn't know I'm named after a *bachata* singer" expressing how there was a need to validate his place within his own culture. These two examples show Anzaldua's point of the borderland, or the liminal space of existence for marginalized cultures in America. It is the idea that one can appear as close to the dominant group as possible and still be excluded for what one expresses of their home culture while simultaneously being excluded from their home culture for what they express of the dominant group.

Another point that influenced which name the participants used came down to respect. One noted that at cultural ceremonies, his name connected him to an extended family with a long oral tradition within his community so when he used his traditional name, it carried more weight. Another commented on how she "felt respected" when someone spoke to her through her traditional name saying "if they're Yoruba too, it makes me feel close to them, if they're not, it makes me feel like they at least care but it can also be too intimate." This reflects how one's name can connect them not just to their culture but to those around them and can also function as a barrier, a way to keep outsiders away from their culture. When asked if they have ever encountered a situation where a person wanted to know their preferred name, participants noted that it does happen but even then, their answer varies depending on the person. Explaining what she meant by "too intimate" the participant said, "I don't know, you can just tell they want to know because they just want to know, not because they want to say it." Another participant had a similar sentiment saying, "sometimes kids would just ask me to say things in Hmong... it made me not want to speak it around them" the main connection between the two and the way it impacted the way they would introduce themselves came down to respect. Both participants acknowledged that in these instances they were being exoticized rather than being approached with genuine respect for their names, languages, and cultures.

Finally, the point that this entire project boils down to is that of intentionally unpacking ethnocentric ideas of language and culture. Throughout the conversations with participants, it was evident that most of these students were never asked to talk about their names before, many had not considered their reasons for choosing one name over the other or why they changed its pronunciation in the first place until these interviews. That is because it has become so normalized, over centuries of forceful assimilation, colonization, and acculturation, for marginalized groups to cater to the white, English-speaking populations at the cost of their own identities. This sacrifice has led to further colonization of the self and resulted in an acculturalization of how people interact with one another. My participants were people that I saw every day, but these interviews did not only give me insight into their names, it taught me about each individual's culture and experience, information that I would have never learned if I did not simply start by asking how to say their names. One participant noted "it's nobody's fault if someone can't say my name, that's fine, but they could at least try" which closely mirrors Anzaldua's words in *Borderlands'* introduction where she affirms her mixture of languages, asking "to be met halfway" (Anzaldua 20). Understanding the historical implications of name changes in America reveals how deeply entrenched ethnocentric ideals are in today's culture and to begin dismantling these structures we have to start by unpacking the ways we contribute to them and making the intentional choice to give others the respect of learning their names.

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Exploring the Lived Experiences of Disability among Hmong Individuals with Disabilities and their Family Members

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Abstract

Understanding the lived experiences of disability among Hmong Americans is important to providing appropriate support for Hmong individuals with disabilities and their family members within the Hmong community and in the wider community, including access to social services. Previous research has focused primarily on parents of individuals with a disability and have not explored the in-depth perspective of other family members, including siblings, and the individuals themselves. Additionally, there is limited research specifically on Hmong individuals with a disability and their families. This research describes a phenomenological study, exploring the lived experiences of disability among five Hmong Americans. Findings from this study highlight key experiences, including the important role of caregivers not only for the individual with a disability but for the entire family; the stigma resulting from negative views and attitudes of disabilities in the Hmong community; and the experiences of accessing social services in the wider community among Hmong families with disabilities.

Introduction

Hmong refugee families began arriving in the United States in the late 1970s, and challenges between Hmong understandings of health and wellbeing and Western medicine have been well documented (Culhane-Pera, et al., 2003). These challenges have created stigma related to obtaining diagnoses (Collier, et al., 2012), barriers to services (Lee & Vang, 2010), and general wariness of health and social services by Hmong individuals (Baker, Dang, et al., 2010; Thorburn, et al., 2012).

One area of health that is important to better understand is the experiences of disability among Hmong individuals with a disability and the experiences of family members of individuals with a disability. In the United States, the term disability has both medical and legal definitions (Lee & Yuen, 2003). Like the definition of disability, experiences of disability also vary, especially when disability is viewed and experienced through a cultural lens. While there is a body of knowledge on the lived experiences of individuals with disabilities and the experiences of parents raising children with disabilities, there is very little research focused specifically on the lived experiences of Hmong individuals living with a disability and the experiences of their family members. The study aims to understand the lived experiences including barriers and stigma Hmong individuals with disabilities and their family members experience both within the Hmong community and the wider community, and access to services. The unique intersectionality of ethnicity and ability provides a better understanding of how disability affects Hmong individuals, their families, and their daily lives.

Literature Review

Our identities are multifaceted and made up of multiple identities and complex experiences. Crenshaw's (1991) concept of intersectionality helps to better understand the idea of overlapping social identities and the experiences of systemic oppression and discrimination. For Hmong individuals, who are already marginalized based on their minoritized identity, they can experience another layer of oppression when they also identify as an individual with a disability.

According to Lee and Yuen (2003), Hmong perceive disability from a spiritual understanding. Two predominant perceptions include disability as a gift to the family, especially when it relates to a child with seizures (they are often viewed as healers), or disability perceived as a punishment for a wrongdoing in a former life (Culhane-Pera, Cha, et al, 2004). These cultural beliefs can make it difficult for some Hmong individuals and families to seek assistance and services, whether within their own communities or outside of Hmong American communities. Other Hmong families may be wary of individuals with disabilities and may even ostracize Hmong individuals with disabilities and their family members. Past research also discussed how Hmong individuals with a disability and their family members felt ashamed to seek help or depressed about their situation, believing they were being punished (Hatmaker, et al., 2010).

Additionally, individuals with disabilities and family members have experienced unsatisfactory support regarding educational services and lack of support for parents to participate in the education of their children (Vang & Barrera, 2004). For example, many students of color with disabilities are often excluded from inclusive educational programs and the general education curriculum (Fierros & Conroy, 2002; LeRoy & Kulik, 2003). And according to the 24th Annual Report to Congress, in 2004, students of color spend 60% or more of their school day in segregated special education placements which are sometimes in schools or classrooms separate from their nondisabled peers. This could place students with disability behind in their classes with their peers and this could affect other factors such as dropping out of school, experiencing high unemployment rates, and lack of preparation for the workforce. Also, most students who are placed into exclusive programs do worse, not better, than they did when they were in mainstream classes (Glass,1983).

For Hmong American parents, another barrier may include limited knowledge of the U.S. educational system. Vang and Barrera (2004) found that Hmong parents did not know how to help their children in education, especially when there was a language barrier, which impacted their involvement and understanding of special education. Most parents appeared more concerned about understanding and communicating basic information about their children's educational progress than providing educators with teaching strategies to meet their children's needs and advocating for those needs.

Lastly, individuals with a disability and their family members express that there are barriers to healthcare services (Baker, Miller, et al., 2010), and have experienced feeling unwelcome in community spaces, agencies, and programs (Heah et al., 2006). Previous research has found that even when individuals of color with developmental disabilities and their families have access to services, they lack accurate information regarding educational and health service delivery and what services their children are receiving. Hmong parents often have limited knowledge of the policies, procedures, practices, and organizational structures of special education (Watham-Ocama & Rose, 2002). Not knowing what services are available can limit their willingness to access services.

Literature also suggests that parents raising a child with a disability experience more stress than parents who raise a child without a disability (Beckman-Bell, 1981; Boyce et al., 1991; Snowdon et al., 1994; Thyen, et al., 1998). The situation can become more incredibly stressful when parents have limited knowledge of systems and have confusion related to medical care and the purpose of appointments or treatments. According to Baker, Miller, et al. (2010), the education and health care systems may not have appropriate mechanisms to address language and cultural barriers. As a result, more information is needed to better understand the experiences of individuals with disabilities and their family members, especially among minoritized groups like Hmong Americans. Understanding these unique lived experiences can better inform not only programs and services, but also understand the experiences of disability within Hmong American communities.

Methods

Phenomenology

This research utilized a qualitative phenomenological design that explores the experiences of the individuals with disabilities and their family members. Phenomenological research designs are "a strategy of inquiry in which the researcher identifies the essence of human experiences about a phenomenon as described by participants" (Creswell, 2009, p 13). Since there is very little research about Hmong individuals with disabilities, a phenomenological design works well to explore and understand the experiences of living with a disability as a Hmong individual and the experiences of raising a child with a disability as a Hmong parent as described by Hmong individuals.

Participants

Family member participants. Five family members between the ages of 21 and 36 years agreed to participate. Four of the family members are siblings of an individual with a disability and one family member is a mother of a child with disability.

Individual with disability participant. There was one individual with a disability who was also part of a family member as a sibling of an individual with disability.

Procedure

This study used purposive sampling techniques. Participants were recruited through contacting social service agencies that serve Hmong clients as well as through social media websites such as Facebook, particularly Facebook groups that have large Hmong audience. Creswell (2009) suggests small sample sizes work best for phenomenological research designs since data collection involves extensive engagement with participants.

In-depth, semi-structured interviews were conducted at private locations which were mutually agreed upon by the participant and researcher. One interview was conducted by phone. Before each interview, participants were given a brief overview of the research and asked to review and sign a consent form. As the interviews were audio taped and transcribed. The interviews were conducted by a research member, and an interview guide was used during the interview (see Appendix A). Each interview lasted approximately 45 to 60 minutes.

Data Analysis

All interviews were transcribed verbatim and analyzed using thematic analysis (Strauss & Corbin, 1990). This analysis involved line-by-line analysis of the interview transcripts, developing levels of codes from general concepts to themes. In addition to the interviews, the researchers drew information from field notes and interview summaries, which were written following each interview to record main thoughts and key points from the interview. The analysis of transcripts, field notes, and interview summaries is a process of triangulation, which helps to cross validate data (Connor, 2008).

Results

Focused themes emerged within four categories: research participants' everyday lives of family members and the individual with disability; perceptions of self, experiences in the Hmong community, and barriers experienced while accessing social services. These themes are presented below.

Everyday Lives

Family members. The study identified the theme of *the everyday lives of the family members*. All of the family members shared that their day centered around their family member with a disability, no matter their relationship, whether a parent or sibling, to the individual. For example, a mother said that she must be flexible with her child's schedule and her work schedule, which can be stressful. The need to have a flexible schedule has impacted the kinds of jobs the mother has been able to obtain. As for siblings, they also must have a flexible schedule to help take care of their family member with a disability. For those siblings in college, this can impact their school schedule and their ability to care for their sibling throughout the day. Others shared that they could not participate in after school activities.

There are also specific duties and responsibilities, including helping with activities of daily living (ADLs). One participant described their typical day with helping their family member with a disability as:

> "a normal typical day [would be] me waking up my older brother [with disability] for the day and to just help him with his necessary activities. Like changing clothes, going to the bathroom, brushing his teeth to washing his face to showering too. Like just your basic necessities that you would need to do."

In addition, the family members also reported that they must assist their family member with a disability with transportation. This includes making sure that transportation is accessible, whether they are providing the transportation or coordinating transportation.

Another important aspect of the everyday lives of family members is assisting with medication. It is essential for the family members to have some knowledge of the medication and the ability to intervene in emergency situations. As one participant shared, "We weren't taught how to help them (family member with disability) directly by professionals but through experience instead." This suggests that there is a lack of professional support or training for family members.

Individual with a Disability. The everyday lived experiences of the individual with disability was different compared to the family members. One major difference is how others interact with the individual with a disability. The one participant with a disability in this study shared that they had an invisible disability. The participant shared that when others find out or know the participant has a disability, people tend to treat the participant differently. The participant stated, "You begin to notice how people treat you once they find out [about the disability]. And it's not apparent, but it is very subtle in how a person behaves towards you and how they talk to you." An individual with a disability has learned to notice and interpret non-verbal cues when they interact with others.

Perceptions of Self

The theme *perception of self* relates to how participants perceive themselves as it relates to disability. Participants shared that they are more understanding of other people with disabilities. Throughout the interviews, participants also shared that they are more empathetic, feel more compassion, and have more understanding for other people because of their specific experiences. One participant said,

> "From a very young age, I've learned to love those that are different than you are and to be more supportive to those who aren't as privileged as you. So, I guess it also taught me patience because having to be patient with them and wait for them to get to places. To just understand things is much more difficult."

For participants, they are not only more aware of the stigma and misconceptions about people with disabilities, but they are more keenly aware of the difficulties and challenges people with disabilities experience.

Experiences in the Hmong Community

This theme includes the experiences of Hmong individuals within the Hmong community. In the Hmong community, there are often negative views and different attitudes towards individuals with a disability. Participants shared that they are often marginalized, and disability is stigmatized, which impacts the entire family. One participant said, "People treat my siblings nicely because they are disabled, and they are not mean to them. But then people just talk like, 'Oh your parents are bad, or you have bad parents because they did something wrong. That's why your siblings are disabled'." So, there is both pity for the individual with the disability and judgement for the family.

Additionally, there is no a direct translation for the word disability in the Hmong language. The most often used term is *ruam*, which translates into English as stupid. This word can have negative conations and meaning that negatively affects the individual with a disability and their family member. The family members interviewed in this study shared that *ruam* is often used by those outside their immediate family to describe their family member with a disability. The participant with a disability said, "The Hmong community acknowledges it (the disability) but they also low-key look down upon you as well because you are seen as a person who is broken and who is not normal." This experience highlights the marginalization of Hmong individuals with a disability within the Hmong community.

Social Services and Barriers

This theme focused on experiences of the participants in terms of access to social services and any barriers they may experience. Most participants reported communication and language barrier as a common reason many do not seek services in the wider community. An example that one participant mentioned was not understanding the medication prescribed to their family member, which made interactions within healthcare settings more challenging. The participant stated, "We are not medical people. So, it is hard to understand what they are talking about sometimes. It's frustrating cause it just feels like, for me, like he is a test subject. Because they don't really know but they just giving him a bunch of stuff but maybe that's how it works. But I don't know." This can be challenging for family members to trust and understand the health professionals when they are not clearly informed about the types of medications and the treatments.

In addition, the participants who took their family member with a disability to health services reported that they had trouble translating some of the terms for specific disabilities and other medical terms to their other Hmong family members who don't speak English. This can make it more difficult for the other family members to understand the disabilities and the services.

There are also some services that do not benefit family members and individuals with a disability in ways that they could. One participant explained, "We had a terrible caseworker who didn't understand and relate to our family life. That Hmong families are different from a White family. This became really stressful." In these cases, it is the cultural competence of service providers and services that make access challenging for Hmong family and individuals with a disability.

Lastly, there is a general lack of awareness related to knowing what resources are available. Not knowing what resources are available makes it harder to know what services and resources to access. A participant shared, "During my late junior or senior year, I learned about a program that I was recommended to, but I wasn't aware of these programs or like these services that were there (at their school). And I can't believe it took me that long in high school to figure it out." These missed opportunities can have great impact on the educational outcomes for students with disabilities.

Discussion

This study describes the lived experiences of five family members of an individual with a disability, including an individual with a disability. The study focused on the lived experiences of Hmong individuals with disability and their family members, exploring how family members and the individual's lives are impacted by disability. In addition, the purpose of this study was to understand the barriers and stigma Hmong individuals with disabilities and their family members experience both within the Hmong community and the wider community.

The data indicated that the roles of the family members were centered around the family member with a disability. Family members often adjust parts of their daily lives in order to take care and meet the ADLs and needs of their family member with a disability. Family members also take on specific duties and have important responsibilities. These duties and responsibilities can impact the kind of jobs and extracurricular activities that family members can participate in, and their responsibilities can have grave consequences if they do not use caution such as giving medication. For family members, these lived experiences can be more stressful in terms of balancing their own life, career or school, and responsibility to their family member with a disability.

While these experiences are stressful, family members do express that through their experience with disability, they tend to be more empathetic toward others and understanding of difference. Their personal experience with disability helps to deconstruct the misconceptions about individuals with disability. Exploring the everyday experiences of the family members is important in understanding how their lives are impacted by disability. For service providers, this understanding assists in thinking about the needs of family members and the strategies for mitigating stress for caregivers.

Although there was only one participant with a disability in this study, there are still important insights into the everyday experiences of disability. As social interactions are part of people's everyday lives, an individual with disability learns to socialize with others differently because of how people may perceive them as different. The participant indicated frustration and having lower confidence from how people treated them. As people have judged the individual with disability immediately without getting a chance to know the individual for who they are as a person and not just as their disability. The stigma around individuals with a disability makes it more difficult for these individuals to have more meaningful interactions with others.

Secondly, this study highlights the importance of intersectionality. Within the Hmong community, there is a stigma related to disability and oftentimes negative attitudes toward disability. This can impact the entire family and the family can be marginalized. One reason it impacts the entire family is that the cause of disability is believed to be spiritual rather than medical. Disability in the Hmong community is associated with karma, superstitions, and spirits. This explains why some Hmong families with experiences of disability are less likely to seek Westernized medicine. These cultural differences are important to know so that non-Hmong professionals can better support and serve their Hmong clients.

Additionally, intersectionality helps to better understand how an individual who is part of a minoritized group is further marginalized when they also have a disability. They are not only marginalized in their own community, but can experience layers of discrimination for being Hmong and having a disability.

Barriers to services also exist among Hmong families with disabilities. Participants in this study shared the linguistic challenges and cultural differences that make seeking services and accessing services difficult. For example, medical terminology is difficult to translate and interpret in the Hmong language. And oftentimes, it is other family members who are serving as interpreters for complex medical problems. This can be hard when family members are not fluent in English, or vice-versa if the interpreter is fluent in English but not in Hmong. Service providers need to be aware of effective communication guidelines, especially working with interpreters.

Additionally, on the part of service users, Hmong clients are not aware of resources available to them. Not aware of resources can result in a belief that there are limited resources in general. There needs to be greater effort to bring resources into communities and to work with communities to create greater awareness. And even if resources do exist, some participants reported lack of access to resources and services. Participants share there was a lack of culturally competent or culturally specific services.

Conclusion

Overall, the study explored key themes about the lived experiences of disability among Hmong Americans, including the daily routines, stigma, and barriers and access to social services. The responses from the interviews revealed that family members living with an individual with a disability are impacted by their role and responsibilities as being a caregiver and provider. This study also further provides insight into the experiences of Hmong individuals and the cultural aspect in how it impacts their perspective on the stigma of disabilities and experiences in the Hmong community. The importance of these experiences can bring more awareness to addressing the misconceptions of disability within the Hmong community. The findings have significant implications for professionals working in social services and for the general public to better understand individuals with disabilities and family members' experiences. It is important to understand the challenges they face from social interactions and how they perceive others and themselves.

While this study had a small sample size and was limited to the perspectives of Hmong individuals, there is still meaningful insight from their experience. Additional research is needed to focus more on the experiences of Hmong individuals with disabilities. Also, more research is needed to understand which services are most effective when working with individuals with disabilities and family members of diverse backgrounds. Recognizing the experiences of disability among Hmong individuals with disabilities and their family members is important to addressing the stigma and barriers Hmong Americans face both within their community and beyond.

Appendix A

Interview Guide: Individual with Disabilities

1. *Can you tell me about yourself?*
    a. *Tell me about your family*
    b. *Do you feel comfortable sharing with me about your disability?*
2. *Tell me about a typical day that you have.*
    a. *Do you face any barriers in your daily activities? What are those?*
3. *What are your experiences as a person with a disability within the Hmong community?*
    a. *Do you face any barriers in this community? What are those?*
    b. *What stigmas related to disability do you think exists within the Hmong community?*
4. *What are your experiences in your community outside of the Hmong community?*
5. *Growing up, were there any programs or special classes you were in during or after school?*
    a. *If so, did they help why or why not?*
6. *Have you or do you currently participate in any services or programs?*
    a. *Have you experienced any barriers within those programs/services?*
7. *Is there anything else you would like to add?*

Interview Guide: Family members of individuals with disabilities

1. *Can you tell me about yourself?*
    a. *Tell me about your family*
2. *What is a typical day like in your family?*
3. *Do you have a role in your family when it comes to your family member (or child) with a disability?*
4. *What have your experiences been with a family member with a disability?*
    a. *How do you think others perceive having a family member (or child) with a disability in the Hmong community?*
    b. *How do you think others perceive having a family member (or child) with a disability in your community outside of the Hmong community?*
5. *Do you face any barriers as a family member (or parent) of a person with a disability?*
6. *Have you or do you currently participate in any services or programs for your family member with a disability?*
    a. *Have you experienced any barriers within those programs/services?*
7. *Is there anything else you would like to add?*

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Fibonacci and Lucas Sequence Identities: Statements and Proofs

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Abstract

This document contains the statements and our own proofs of an enormous array of identities related to the Fibonacci sequence. Most of the identities were taken either from the book *Fibonacci Numbers* by Nicolai Vorobiev [4] and *Elementary Number Theory* by David Burton [1].

Contents

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1 Preliminaries and Definitions

The Fibonacci sequence $(F_n)_{n=0}^{\infty}$ is ubiquitous in mathematics and has been a well-studied object since appearing in the text *Liber abaci* in the year 1202 by the renowned Italian mathematician Leonardo Pisano Bigollo (more commonly known today by the name "Fibonacci" coined by French historian Guillaume Libri in 1838).

Definition 1.1. The *Fibonacci sequence* $(F_n)_{n=0}^{\infty}$ is defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2} \tag{1}$$

with initial conditions $F_0 = 0$ and $F_1 = 1$. This sequence has the following closed form

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n. \tag{2}$$

Observing the *golden ratio* ϕ is the value $\frac{1+\sqrt{5}}{2}$ and the value $\frac{1-\sqrt{5}}{2}$ equals $1-\phi = -\frac{1}{\phi} = -\phi^{-1}$, we may rewrite Equation (2) in the following compact expression $F_n = \frac{1}{\sqrt{5}}\left(\phi^n - (-\phi)^{-n}\right)$.

2 List of Every Statement Proven in this Paper

Chapter 1 Identities from Vorobiev

1. **Proposition 4.1** (Identity (1.1) [4]). For all $n \geq 1$, we have
$$F_1 + F_2 + \cdots + F_n = F_{n+2} - 1.$$

2. **Proposition 4.2** (Identity (1.2) [4]). For all $n \geq 1$, we have
$$F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}.$$

3. **Proposition 4.3** (Identity (1.3) [4]). For all $n \geq 1$, we have
$$F_2 + F_4 + F_6 + \cdots + F_{2n} = F_{2n+1} - 1.$$

4. **Lemma 4.4** (so-called "Fibberish" Identity). The following are equivalent to the Fibonacci recurrence $F_n = F_{n-1} + F_{n-2}$:
$$F_{n-1} = F_n - F_{n-2} \tag{3}$$
$$F_{n-2} = F_n - F_{n-1} \tag{4}$$

5. **Proposition 4.5** (Identity (1.4) [4]). For all $n \geq 1$, we have
$$F_1 - F_2 + F_3 - F_4 + \cdots + F_{2n-2} + F_{2n-1} - F_{2n} = -F_{2n-1} + 1.$$

6. **Proposition 4.6** (Identity (1.5) [4]). For all $n \geq 1$, we have
$$F_1 + F_2 + F_3 + F_4 + \cdots + F_{2n} + F_{2n+1} = F_{2n} + 1.$$

7. **Proposition 4.7** (Identity (1.6) [4]). For all $m \geq 1$, we have
$$F_1 - F_2 + F_3 - F_4 + \cdots + (-1)^{m+1}F_m = (-1)^{m+1}F_{m-1} + 1.$$

8. **Proposition 4.8** (Identity (1.7) [4]). For all $m \geq 1$, we have
$$F_1^2 + F_2^2 + \cdots + F_n^2 = F_nF_{n+1}.$$

9. **Proposition 4.10** (Identity (1.8) [4]). For all $m \geq 1$ and n fixed as a natural number, we have
$$F_{n+m} = F_{n-1}F_m + F_nF_{m+1}.$$

10. **Corollary 4.11** (Corollary 1 of Proposition 4.10). For all $n \geq 1$, we have
$$F_{2n} = F_{n-1}F_n + F_nF_{n+1}.$$

11. **Corollary 4.12** (Corollary 2 of Proposition 4.10). For all $n \geq 1$, we have

$$F_{2n} = F_{n+1}^2 - F_{n-1}^2.$$

12. **Remark 4.13**. From Corollary 4.11, an equivalent expression for F_{2n} is

$$F_{2n} = F_n(F_{n-1} + F_{n+1}).$$

13. **Corollary 4.14** (Corollary 3 of Proposition 4.10). For all $n \geq 1$, we have

$$F_{2n-1} = F_{n-1}^2 + F_n^2.$$

14. **Corollary 4.15** (Corollary 4 of Proposition 4.10). For all $n \geq 1$, we have

$$F_{3n} = F_{n+1}^3 + F_n^3 - F_{n-1}^3.$$

15. **Proposition 4.16** (Cassini's Identity, also Identity (1.10) [4]) For all $n \geq 2$, we have

$$F_n^2 = F_{n+1}F_{n-1} + (-1)^{n+1}.$$

Golden Ratio, Binet's Formula, and Lucas Numbers

16. **Proposition 5.1** (Sharon, jus a gal nextdoor). For all $n \geq 1$, we have

$$\phi^n = \phi F_n + F_{n-1}.$$

17. **Proposition 5.2**. The n^{th} Fibonacci number can be calculated by the following:

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

18. **Lemma 5.6**. Let α and β be defined as follows

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2}.$$

Then the following equalities hold:

$$\alpha + \beta = 1 \tag{8}$$
$$\alpha - \beta = \sqrt{5} \tag{9}$$
$$\alpha\beta = -1 \tag{10}$$
$$\alpha^2 = \alpha + 1 \tag{11}$$
$$\beta^2 = \beta + 1 \tag{12}$$

19. **Proposition 5.5**. The Lucas sequence has the following closed form

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Compactly, we write $L_n = \alpha^n + \beta^n$ where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

20. **Lemma 5.4** (so-called "Lucarish" Identity). The following are equivalent to the Lucas recurrence $L_n = L_{n-1} + L_{n-2}$:

$$L_{n-1} = L_n - L_{n-2} \tag{6}$$
$$L_{n-2} = L_n - L_{n-1} \tag{7}$$

21. **Proposition 5.7**. For all $n \geq 1$, we have $F_{n+2}^2 - F_n^2 = F_{2n+2}$.

22. **Proposition 5.8**. For all $n \geq 1$, we have $F_{2n+1}F_{2n-1} - 1 = F_{2n}^2$.

GCD and Divisibility Properties and Well-Known Congruence Theorems

23. **Lemma 6.1**. If $b \mid c$, then $\gcd(a+c, b) = \gcd(a, b)$.

24. **Lemma 6.2**. If $\gcd(a, c) = 1$, then $\gcd(a, bc) = \gcd(a, b)$.

25. **Lemma 6.3** (Bézout's Identity). If $\gcd(a, b) = 1$, then there exists $x, y \in \mathbb{Z}$ such that $1 = ax + by$.

26. **Lemma 6.4**. If $a \mid n$ and $b \mid n$ with $\gcd(a, b) = 1$, then $ab \mid n$.

27. **Lemma 6.6**. If $a \mid bc$ and $\gcd(a, b) = 1$, then $a \mid c$.

28. **Theorem 6.7** (Wilson's Theorem). If p is a prime, then $(p-1)! \equiv -1 \pmod{p}$.

Fibonacci Identities from Burton in Section 14.2

29. **Proposition 7.6** For all $n \geq 1$, the following identity holds:

$$\gcd(F_n, F_{n+1}) = 1.$$

30. **Proposition 7.2**. For $m, n \geq 1$, we have F_{mn} is divisible by F_m.

31. **Lemma 7.3**. If $m = qn + r$, then $\gcd(F_m, F_n) = \gcd(F_r, F_n)$.

32. **Proposition 7.4**. The greatest common divisor of two Fibonacci numbers is again a Fibonacci number. Specifically, we have

$$(F_m, F_n) = F_d \text{ where } d = \gcd(m, n).$$

33. **Corollary 7.5**. For all $n \geq m \geq 3$, we have $F_m \mid F_n$ if and only if $m \mid n$.

34. **Claim 7.7** (Exercise 1 from 14.2 [1]). Given prime $p \neq 5$, then either F_{p-1} or F_{p+1} is divisible by p. Confirm this in the cases of the primes 7, 11, 13, and 17.

35. **Proposition 7.8** (Exercise 2 from 14.2 [1]). For $n = 1, 2, \ldots 10$, show that $F_n^2 + 4(-1)^n$ is always a perfect square.

36. **Proposition 7.9** (Exercise 3 from 14.2 [1]). If $2 \mid F_n$, then $4 \mid F_{n+1}^2 - F_{n-1}^2$. And similarly, if $3 \mid F_n$ then $9 \mid F_{n+1}^3 - F_{n-1}^3$.

37. **Proposition 7.10** (Exercise 4 from 14.2 [1]).

    (a) $F_{n+3} \equiv F_n \pmod 2$, and

    (b) $F_{n+5} \equiv 3F_n \pmod 5$.

38. **Proposition 7.11** (Exercise 5 from 14.2 [1]). $F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}$.

39. **Proposition 7.13** (Exercise 6 from 14.2 [1]). For all $n \geq 3$, we have

$$F_{n+1}^2 = F_n^2 + 3F_{n-1}^2 + 2\big(F_{n-2}^2 + F_{n-3}^2 + \cdots + F_2^2 + F_1^2\big).$$

40. **Proposition 7.14** (Exercise 7 from 14.2 [1]). Evaluate $\gcd(F_9, F_{12}), \gcd(F_{15}, F_{20})$and $\gcd(F_{24}, F_{36})$.

41. **Proposition 7.15** (Exercise 8 from 14.2 [1]). Find the Fibonacci numbers that divide both F_{24} and F_{36}

42. **Proposition 7.16** (Exercise 9 from 14.2 [1]). Use the fact that $F_m \mid F_n$ if and only if $m \mid n$ (i.e., Corollary 7.5) to verify each assertion below:

    (a) $2 \mid F_n$ if and only if $3 \mid n$.

    (b) $3 \mid F_n$ if and only if $4 \mid n$.

    (c) $5 \mid F_n$ if and only if $5 \mid n$.

    (d) $8 \mid F_n$ if and only if $6 \mid n$.

43. **Proposition 7.17** (Exercise 10 from 14.2 [1]). If $\gcd(m, n) = 1$, then $F_m F_n$ divides F_{mn} for all $m, n \geq 1$.

44. **Proposition 7.18** (Exercise 11 from 14.2 [1]). It can be shown that when F_n is divided by F_m where $n > m$, then the remainder r is a Fibonacci number or $F_m - r$ is a Fibonacci number. Give examples illustrating both cases.

45. **Proposition 7.19** (Exercise 12 from 14.2 [1]). It was proven in 1989 that ther are only five Fibonacci numbers that are also triangular numbers. Find them.

46. **Proposition 7.21** (Exercise 13 from 14.2 [1]). If $n \geq 1$ then $2^{n-1}F_n \equiv n \pmod 5$.

47. **Proposition 7.22** (Exercise 14 from 14.2 [1]). If $F_n < a < F_{n+1} < b < F_{n+2}$ for some $n \geq 4$ establish that the sum $a + b$ cannot be a Fibonacci number.

48. **Proposition 7.23** (Exercise 15 from 14.2 [1]). There is no positive integer n for which $F_1 + F_2 + F_3 + \cdots + F_{3n} = 16!$

49. **Lemma 7.25**. If $n \in 3\mathbb{Z} + 1$ or $n \in 3\mathbb{Z} + 2$, then F_n is odd.

50. **Proposition 7.26** (Exercise 16 from 14.2 [1]). If 3 divides $n + m$, show that

$$F_{n-m-1}F_n + F_{n-m}F_{n+1}$$

is an even integer.

51. **Proposition 7.27** (Exercise 17 from 14.2 [1]). For all $n \geq 1$, Verify that there exist n consecutive composite Fibonacci numbers.

52. **Proposition 7.29** (Exercise 18 from 14.2 [1]). $9 \mid F_{n+24}$ if and only if $9 \mid F_n$.

53. **Proposition 7.30** (Exercise 19 from 14.2 [1]). If $n \geq 1$ then $F_{2n} \equiv n(-1)^{n+1}$.

54. **Proposition 7.31** (Exercise 20 from 14.2 [1]). Derive the identity $F_{n+3} = 3F_{n+1} - F_{n-1}$ for all $n \geq 2$.

Fibonacci Identities from Burton in Section 14.3

55. **Proposition 7.32** (Exercise 3a from 14.3 [1]). For all $n \geq 2$, the following identity holds:

$$F_{2n-1} = F_n^2 + F_{n-1}^2.$$

56. **Proposition 7.33** (Exercise 3b from 14.3 [1]). For all $n \geq 2$, the following identity holds:

$$F_{2n} = F_{n+1}^2 - F_{n-1}^2.$$

57. **Proposition 7.35** (Exercise 4a from 14.3 [1]). For all $n \geq 3$, the following identity holds:

$$F_{n+1}^2 + F_{n-2}^2 = 2F_{2n-1}.$$

58. **Proposition 7.36** (Exercise 4b from 14.3 [1]). For all $n \geq 2$, the following identity holds:
$$F_{n+2}^2 + F_{n-1}^2 = 2(F_n^2 + F_{n+1}^2).$$

59. **Proposition 7.37** (Exercise 6a from 14.3 [1]). For all $n \geq 2$, the following identity holds:
$$F_{n+1}^2 - 4F_nF_{n-1} = F_{n-2}^2.$$

60. **Proposition 7.38** (Exercise 6b from 14.3 [1]). For all $n \geq 3$, the following identity holds:
$$F_{n+1}F_{n-1} - F_{n+2}F_{n-2} = 2(-1)^n.$$

61. **Proposition 7.39** (Exercise (6c) from 14.3 [1]2). For all $n \geq 3$, the following identity holds:
$$F_n^2 - F_{n+2}F_{n-2} = (-1)^n.$$

62. **Proposition 7.40** (Exercise 6e from 14.3 [1]). For all $n \geq 1$, the following identity holds:
$$F_nF_{n+1}F_{n+3}F_{n+4} = F_{n+2}^4 - 1.$$

63. **Proposition 7.41**(Exercise 11 from 14.3 [1]) For all $n \geq 1$, the following identity holds:
$$F_{2n+2}F_{2n-1} - F_{2n}F_{2n+1} = 1.$$

64. **Proposition 7.42** (Exercise 15 from 14.3 [1]). Prove that the sum of any 20 consecutive Fibonacci numbers is divisible by F_{10}.

65. **Proposition 7.43** (Exercise 16 from 14.3 [1]) For all $n \geq 4$, the number $F_n + 1$ is not prime. In particular, the following four identities hold:
$$\begin{aligned} F_{4k} + 1 &= F_{2k-1}(F_{2k} + F_{2k+2}) \\ F_{4k+1} + 1 &= F_{2k+1}(F_{2k-1} + F_{2k+1}) \\ F_{4k+2} + 1 &= F_{2k+2}(F_{2k+1} + F_{2k-1}) \\ F_{4k+3} + 1 &= F_{2k+1}(F_{2k+1} + F_{2k+3}). \end{aligned}$$

66. **Proposition 7.45** (Exercise 23 from 14.3 [1]). The following gives a formula for the Fibonacci numbers in terms of the binomial coefficients:
$$F_n = \binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \cdots + + \binom{n-j}{j-1} + \binom{n-j-1}{j}.$$

Lucas Identities from Burton in Section 14.3

67. **Proposition 8.1** (Exercise 17a from 14.3 [1]). For all $n \geq 1$, the following identity holds:
$$L_1 + L_2 + L_3 + \cdots + L_n = L_{n+2} - 3.$$

68. **Proposition 8.2** (Exercise 17b from 14.3 [1]). For all $n \geq 1$, the following identity holds:
$$L_1 + L_3 + L_5 + \cdots + L_{2n-1} = L_{2n} - 2.$$

69. **Proposition 8.3** (Exercise 17c from 14.3 [1]) For all $n \geq 1$, the following identity holds:
$$L_2 + L_4 + L_6 + \cdots + L_{2n} = L_{2n+1} - 1.$$

70. **Proposition 8.4** (Exercise 17d from 14.3 [1]). For all $n \geq 1$, the following identity holds:
$$L_n^2 = L_{n+1}L_{n-1} + 5(-1)^n.$$

71. **Proposition 8.5** (Exercise 17e from 14.3 [1]). For all $n \geq 1$, the following identity holds:
$$L_1^2 + L_2^2 + L_3^2 + \cdots + L_n^2 = L_n L_{n+1} - 2.$$

72. **Proposition 8.6** (Exercise 17f from 14.3 [1]). For all $n \geq 2$, the following identity holds:
$$L_{n+1}^2 - L_n^2 = L_{n-1}L_{n+2}.$$

73. **Proposition 8.8** (Exercise 18a from 14.3 [1]). For all $n \geq 1$, the following identity holds:
$$L_n = F_{n+1} + F_{n-1} = F_n + 2F_{n-1}.$$

74. **Proposition 8.9** (Exercise 18b from 14.3 [1]). For all $n \geq 3$, the following identity holds:
$$L_n = F_{n+2} - F_{n-2}.$$

75. **Proposition 8.10** (Exercise 18c from 14.3 [1]). For all $n \geq 1$, the following identity holds:
$$F_{2n} = F_n L_n.$$

76. **Proposition 8.11** (Exercise 18d from 14.3 [1]) For all $n \geq 2$, the following identity holds:
$$L_{n+1} + L_{n-1} = 5F_n.$$

77. **Proposition 8.12** (Exercise 18e from 14.3 [1]). For all $n \geq 2$, the following identity holds:
$$L_n^2 = F_n^2 - 4F_{n+1}F_{n-1}.$$

78. **Proposition 8.13** (Exercise 18f from 14.3 [1]) For all $n, m \geq 1$, the following identity holds:
$$2F_{m+n} = F_m L_n + L_m F_n.$$

79. **Proposition 8.14** (Exercise 18g from 14.3 [1]) For all $n, m \geq 1$, the following identity holds:
$$\gcd(F_n, L_n) = 1 \text{ or } 2.$$

80. **Proposition 8.15** (Exercise 20a from 14.3 [1]). For all $n \geq 1$, the following identity holds:
$$L_n^2 = L_{2n} + 2(-1)^n.$$

81. **Proposition 8.16** (Exercise 20b from 14.3 [1]). For all $n \geq 1$, the following identity holds:
$$L_n L_{n+1} - L_{2n+1} = (-1)^n.$$

82. **Proposition 8.17** (Exercise 20c from 14.3 [1]). For all $n \geq 2$, the following identity holds:
$$L_n^2 - L_{n-1}L_{n+1} = 5(-1)^n.$$

83. **Proposition 8.18** (Exercise 20d from 14.3 [1]). For all $n \geq 3$, the following identity holds:
$$L_{2n} + 7(-1)^n = L_{n-2}L_{n+2}.$$

84. **Proposition 8.19** (Exercise 21a from 14.3 [1]). For all $n \geq 1$, the following identity holds:
$$L_n^2 - 5F_n^2 = 4(-1)^n.$$

85. **Proposition 8.20** (Exercise 21b from 14.3 [1]). For all $n \geq 1$, the following identity holds:
$$L_{2n+1} = 5F_n F_{n+1} + (-1)^n.$$

86. **Proposition 8.21** (Exercise 21c from 14.3 [1]). For all $n \geq 2$, the following identity holds:
$$L_n^2 - F_n^2 = 4F_{n+1}F_{n-1}.$$

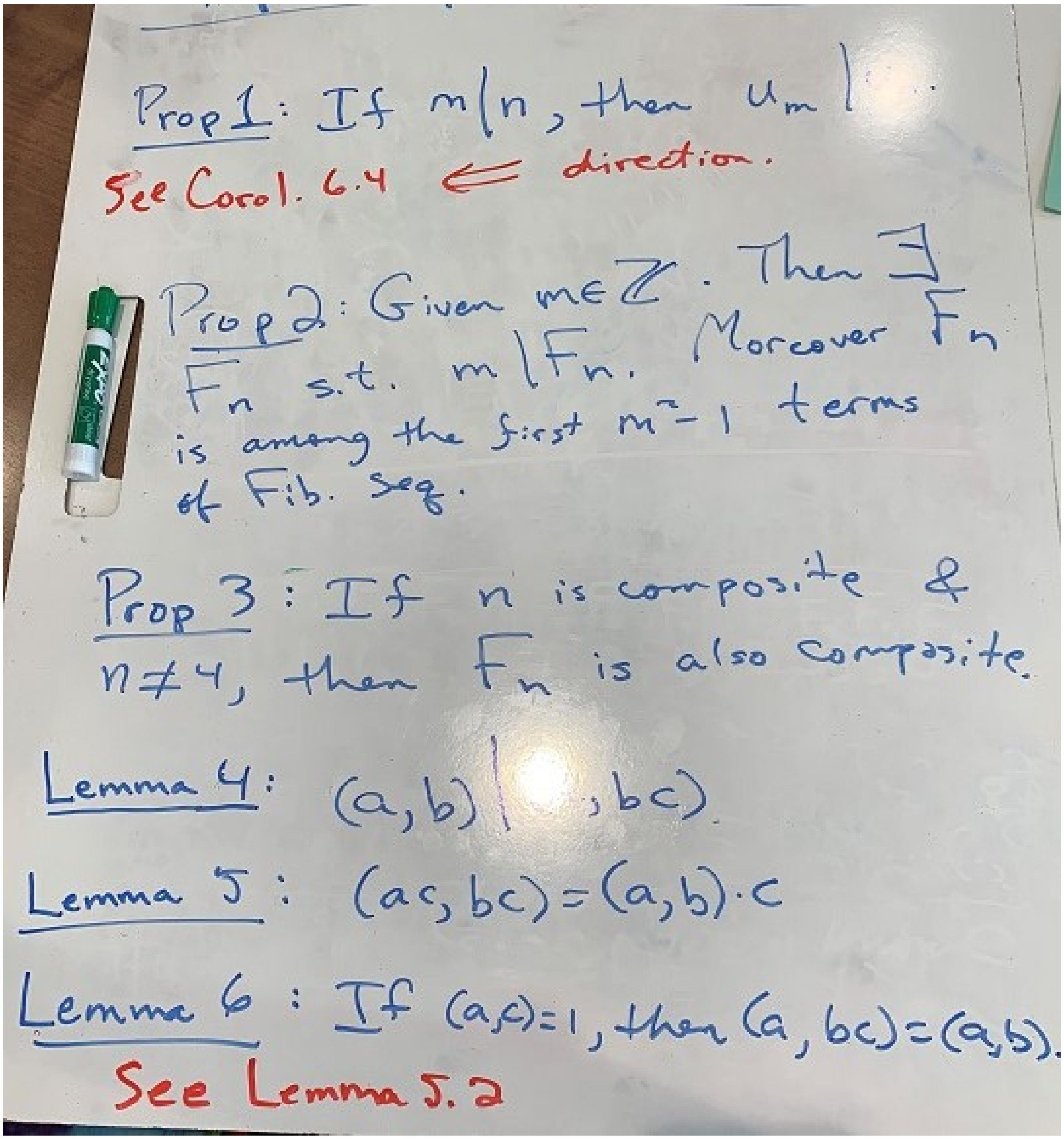
87. **Proposition 8.22** (Exercise 21d from 14.3 [1]) Use the Binet formulas to obtain the relations below:
$$L_m L_n + 5F_m F_n = 2L_{m+n}.$$

88. **Proposition 8.23** (Exercise 22 from 14.3 [1]) For all $n \geq 2$, the following identity holds:
$$L_{2^n} \equiv 7 \pmod{10}.$$

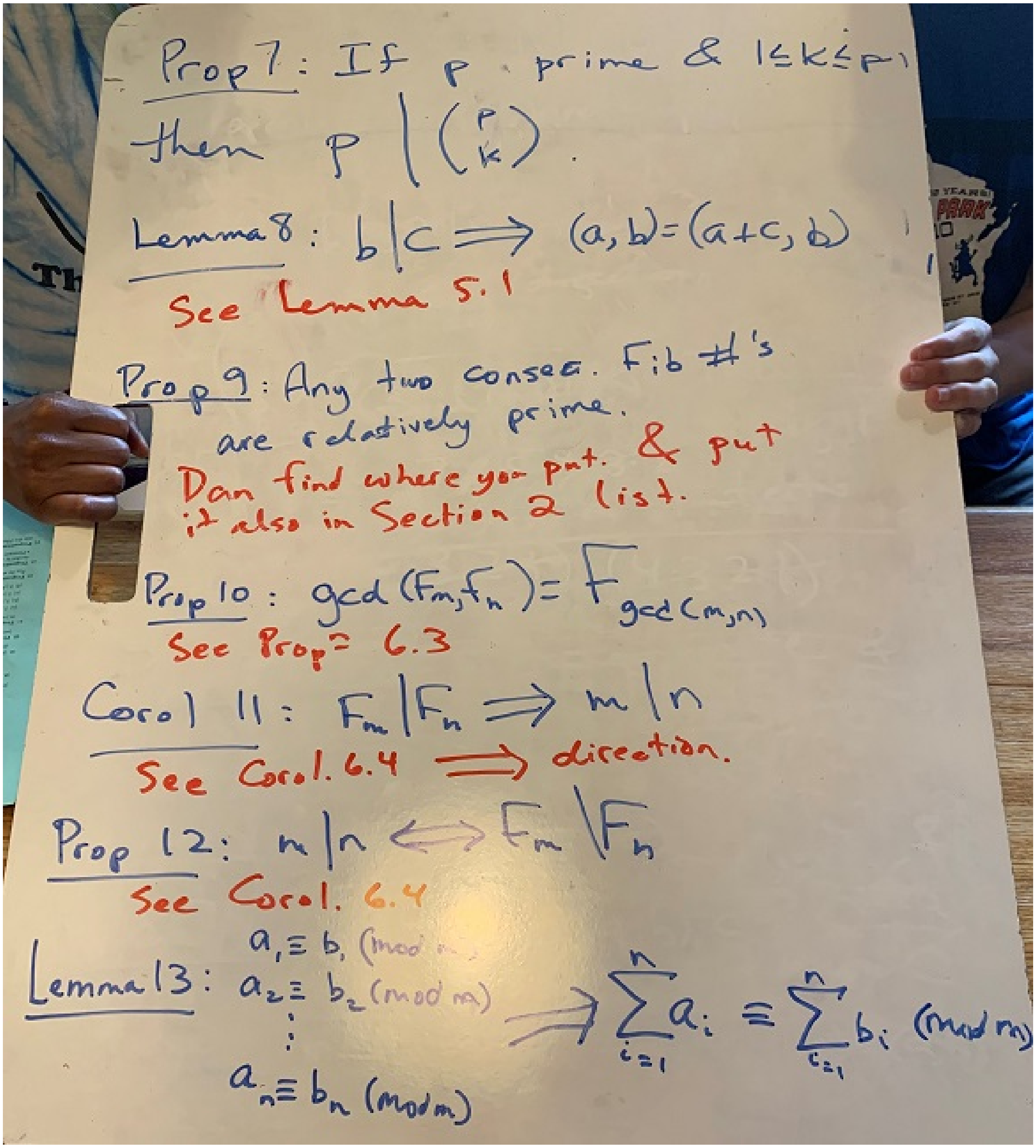
Number Theoretic Properties of the Fibonacci Sequence from Vorobiev Chapter 2

The most important result below has many implications to understanding the Fibonacci sequence modulo m.

Proposition 2.1. *Given $m \in \mathbb{Z}$. Then there exists an F_n such that $m \mid F_n$. Moreover, F_n is among the first $m^2 - 1$ terms of the Fibonacci sequence.*



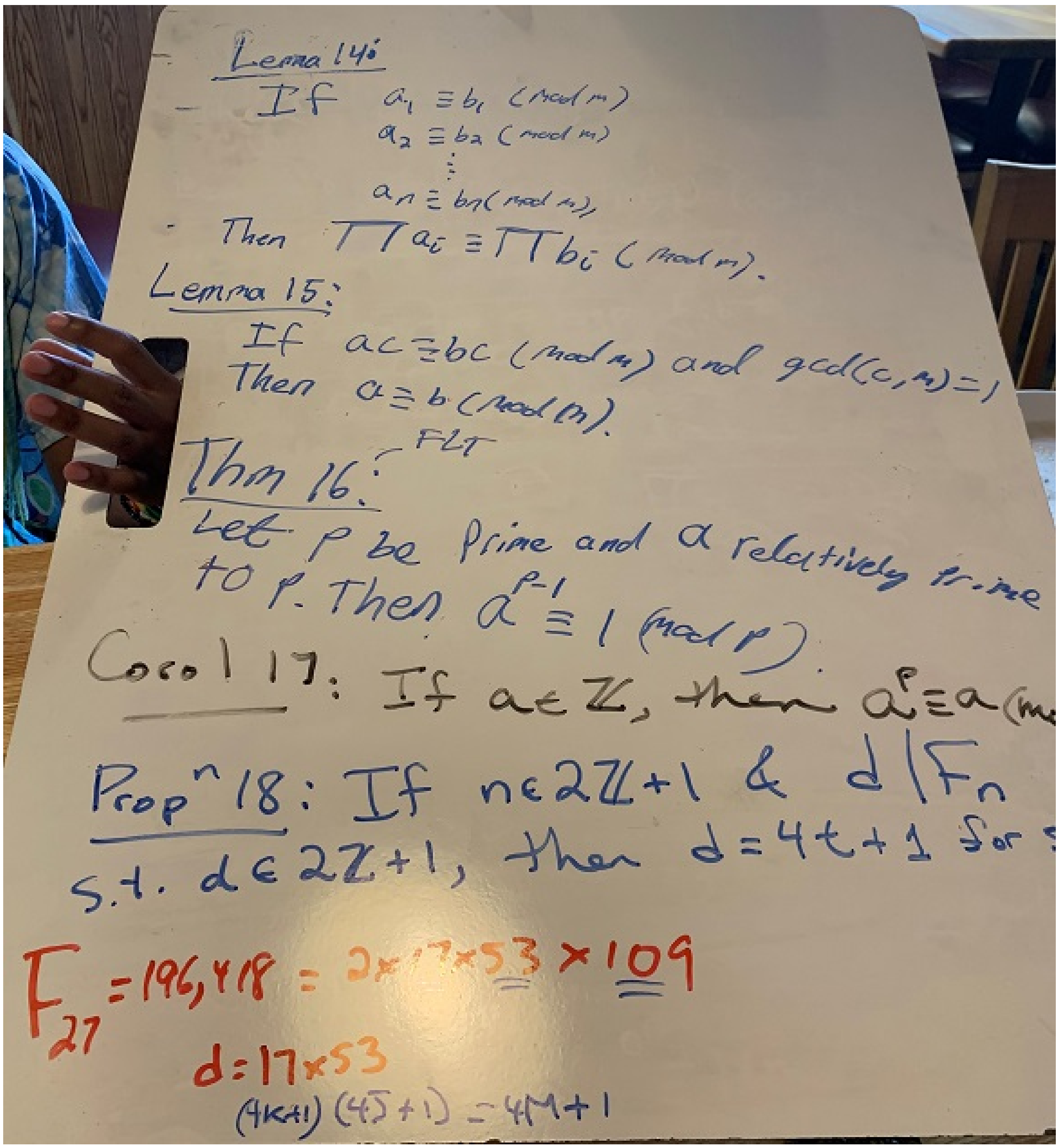
Most of the results below have been proven by us in various parts of this paper. And if not, then they are well-known result in elementary number thoery.



Of the statements below, the most important is Fermat's Little Theorem. And the proposition that follows is a most intriguing result regarding odd divisors of Fibonacci numbers with odd index.

Theorem 2.2 (Fermat's Little Theorem)**.** *Let p be a prime and suppose that* $\gcd(a, p) = 1$. *Then* $a^{p-1} \equiv 1 \pmod{p}$.

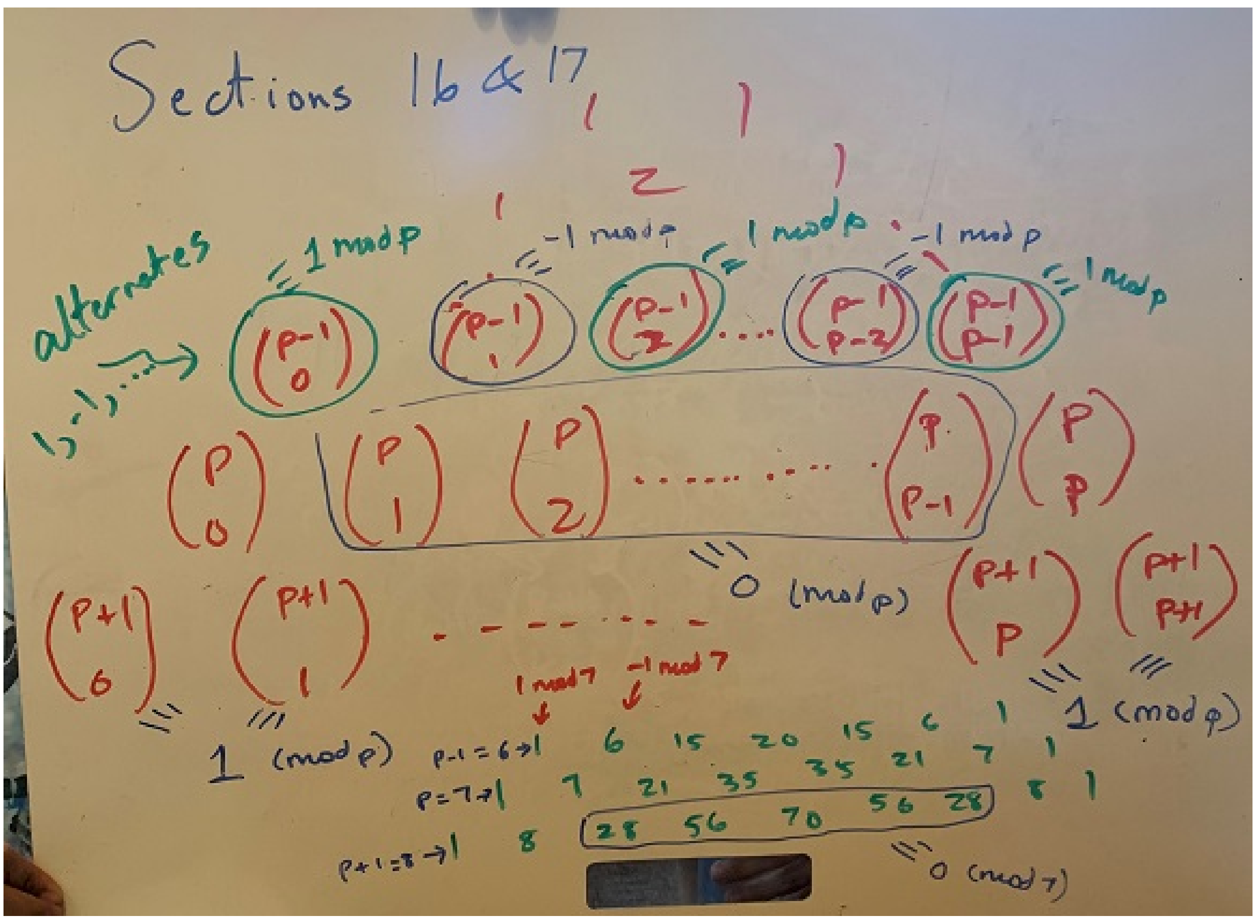
Proposition 2.3. *If an odd integer d is a divisor of F_n where n is also odd, then d if of the form $4t + 1$ for some $t \in \mathbb{Z}$.*



Proposition 2.4. *Let p be a prime. Then in Pascal's triangle, the following holds:*

- *All the binomial coefficients in the $(p-1)^{\text{th}}$ row alternate between being 1 and -1 modulo p.*
- *All the binomial coefficients in the p^{th} row (except the first and last) are divisible by p. Moreover, the first and last coefficients are both congruent to 1 modulo p.*
- *All the binomial coefficients in the $(p+1)^{\text{th}}$ row (except the first two and last two) are divisible by p. Moreover, these first two and last two coefficients are all congruent to 1 modulo p.*

The image below illustrates the proposition above. In green color at the bottom of the image, we give a concrete example with $p = 7$.



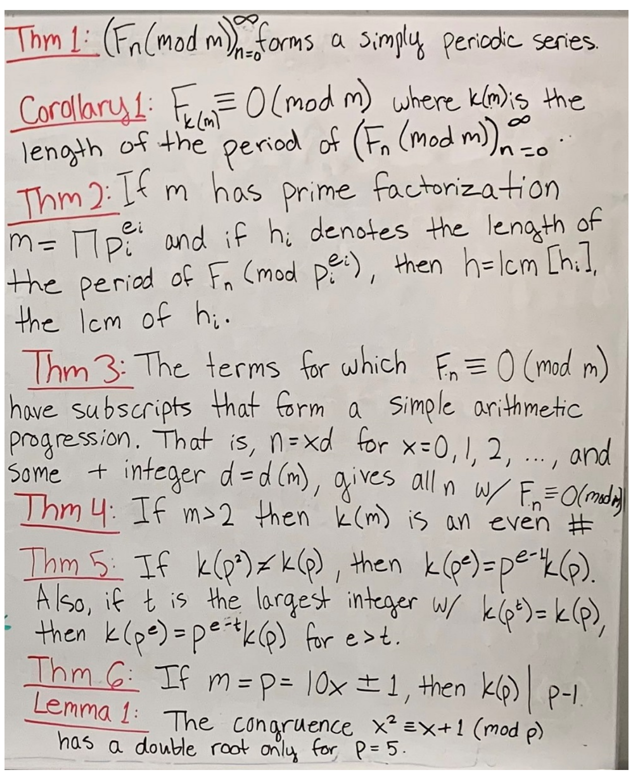
3 Results from Wall, Vinson, and Vince

In this section we collect the main results from the three papers by Wall [5], Vinson [3], and Vince [2], respectively. Each uses the same concepts but do not coincide with symbols used to denote them. Below we give a table that collects this data.

	Wall (1960)	**Vinson (1963)**	**Vince (1978)**
Period of $(\mathcal{F}_{m,n})_{n=0}^{\infty}$	$k(m)$	$s(m)$	$\sigma(m)$
Rank of apparition of $(\mathcal{F}_{m,n})_{n=0}^{\infty}$	not sure	$f(m)$	$\rho(m)$
$\dfrac{\text{period of } (\mathcal{F}_{m,n})_{n=0}^{\infty}}{\text{rank of apparition of } (\mathcal{F}_{m,n})_{n=0}^{\infty}}$	$d(m)$	$t(m)$	$D(m)$

Wall defines $h(m)$ to the be the period of the **generalized Fibonacci sequence** $(G_n)_{n=0}^{\infty}$ where $G_n = G_{n-1} + G_{n-2}$ for $n \geq 2$ with initial conditions $G_0 = a$ and $G_1 = b$ for some integers $a, b \in \mathbb{Z}$.

Results from Wall (1960)


Thm 1: $(F_n (\text{mod } m))_{n=0}^{\infty}$ forms a simply periodic series.

Corollary 1: $F_{k(m)} \equiv 0 (\text{mod } m)$ where $k(m)$ is the length of the period of $(F_n (\text{mod } m))_{n=0}^{\infty}$.

Thm 2: If m has prime factorization $m = \prod p_i^{e_i}$ and if h_i denotes the length of the period of $F_n (\text{mod } p_i^{e_i})$, then $h = \text{lcm}[h_i]$, the lcm of h_i.

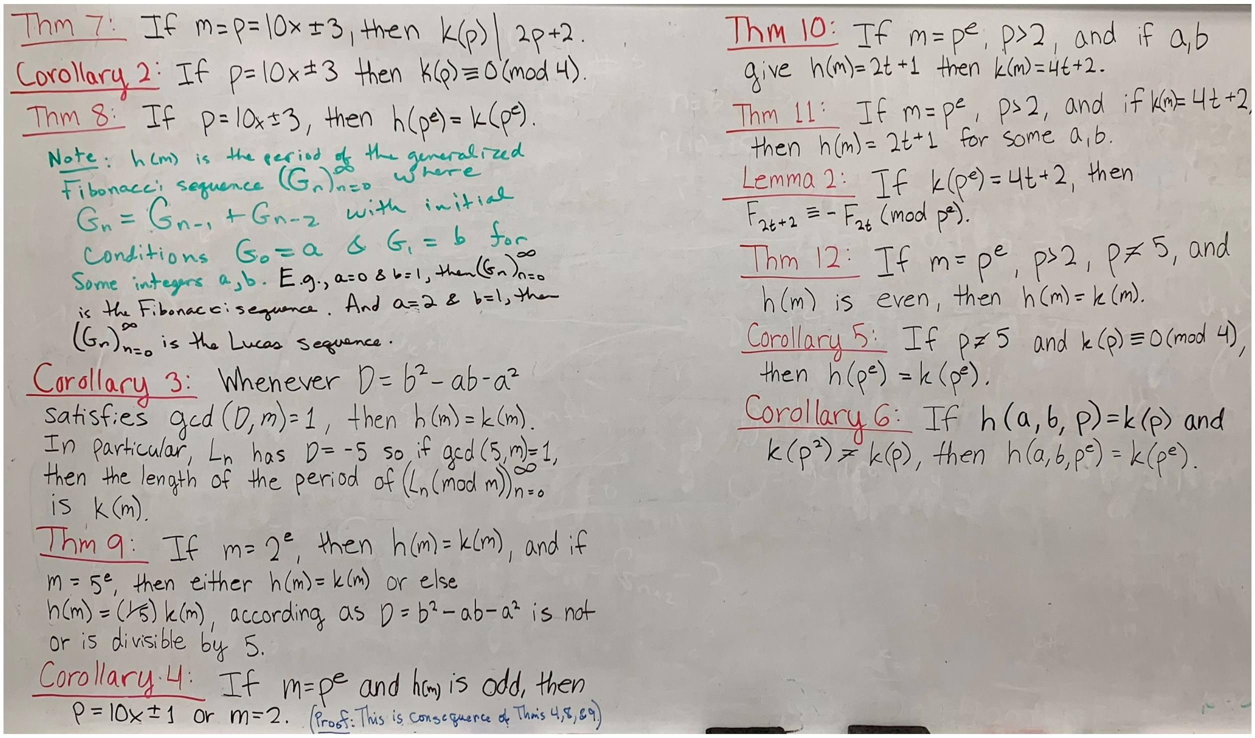
Thm 3: The terms for which $F_n \equiv 0 (\text{mod } m)$ have subscripts that form a simple arithmetic progression. That is, $n = xd$ for $x = 0, 1, 2, \dots,$ and some + integer $d = d(m)$, gives all n w/ $F_n \equiv 0 (\text{mod } m)$

Thm 4: If $m > 2$ then $k(m)$ is an even #

Thm 5: If $k(p^2) \neq k(p)$, then $k(p^e) = p^{e-1} k(p)$. Also, if t is the largest integer w/ $k(p^t) = k(p)$, then $k(p^e) = p^{e-t} k(p)$ for $e > t$.

Thm 6: If $m = p = 10x \pm 1$, then $k(p) \mid p-1$

Lemma 1: The congruence $x^2 \equiv x + 1 (\text{mod } p)$ has a double root only for $p = 5$.



Thm 7: If $m=p=10x\pm3$, then $k(p) \mid 2p+2$.

Corollary 2: If $p=10x\pm3$ then $k(p)\equiv 0 \pmod 4$.

Thm 8: If $p=10x\pm3$, then $h(p^e)=k(p^e)$.

Note: $h(m)$ is the period of the generalized Fibonacci sequence $(G_n)_{n=0}^{\infty}$ where $G_n=G_{n-1}+G_{n-2}$ with initial conditions $G_0=a$ & $G_1=b$ for some integers a,b. E.g., $a=0$ & $b=1$, then $(G_n)_{n=0}^{\infty}$ is the Fibonacci sequence. And $a=2$ & $b=1$, then $(G_n)_{n=0}^{\infty}$ is the Lucas sequence.

Corollary 3: Whenever $D=b^2-ab-a^2$ satisfies $\gcd(D,m)=1$, then $h(m)=k(m)$. In particular, L_n has $D=-5$ so if $\gcd(5,m)=1$, then the length of the period of $(L_n \pmod m)_{n=0}^{\infty}$ is $k(m)$.

Thm 9: If $m=2^e$, then $h(m)=k(m)$, and if $m=5^e$, then either $h(m)=k(m)$ or else $h(m)=(1/5)k(m)$, according as $D=b^2-ab-a^2$ is not or is divisible by 5.

Corollary 4: If $m=p^e$ and $h(m)$ is odd, then $p=10x\pm1$ or $m=2$. (Proof: This is consequence of Thms 4,8,&9)

Thm 10: If $m=p^e$, $p>2$, and if a,b give $h(m)=2t+1$ then $k(m)=4t+2$.

Thm 11: If $m=p^e$, $p>2$, and if $k(m)=4t+2$, then $h(m)=2t+1$ for some a,b.

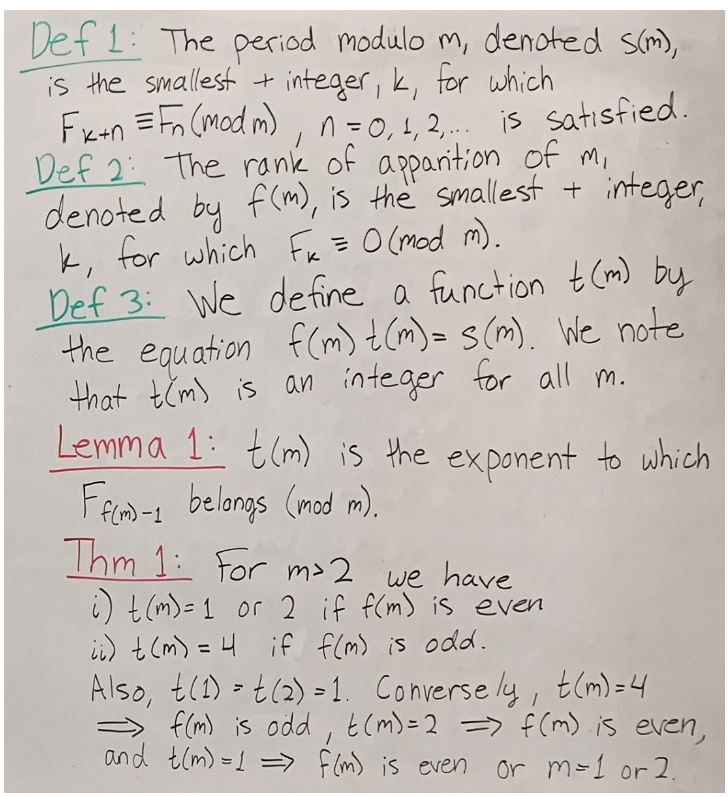
Lemma 2: If $k(p^e)=4t+2$, then $F_{2t+2}\equiv -F_{2t} \pmod{p^e}$.

Thm 12: If $m=p^e$, $p>2$, $p\neq 5$, and $h(m)$ is even, then $h(m)=k(m)$.

Corollary 5: If $p\neq 5$ and $k(p)\equiv 0 \pmod 4$, then $h(p^e)=k(p^e)$.

Corollary 6: If $h(a,b,p)=k(p)$ and $k(p^2)\neq k(p)$, then $h(a,b,p^e)=k(p^e)$.

Results from Vinson (1963)



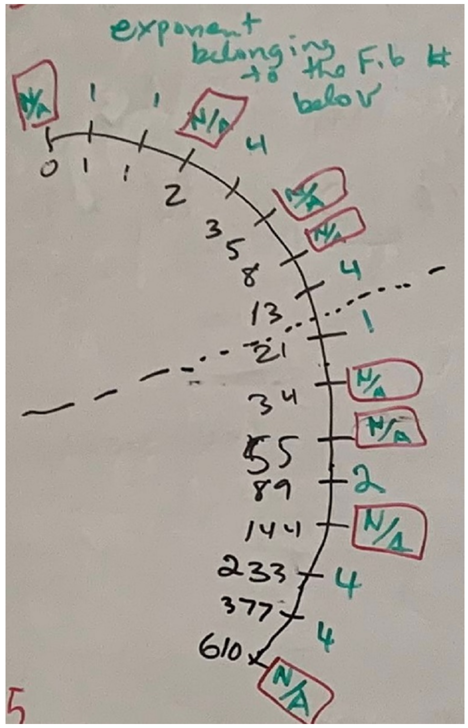
Def 1: The period modulo m, denoted $s(m)$, is the smallest + integer, k, for which $F_{k+n}\equiv F_n \pmod m$, $n=0,1,2,\ldots$ is satisfied.

Def 2: The rank of apparition of m, denoted by $f(m)$, is the smallest + integer, k, for which $F_k\equiv 0 \pmod m$.

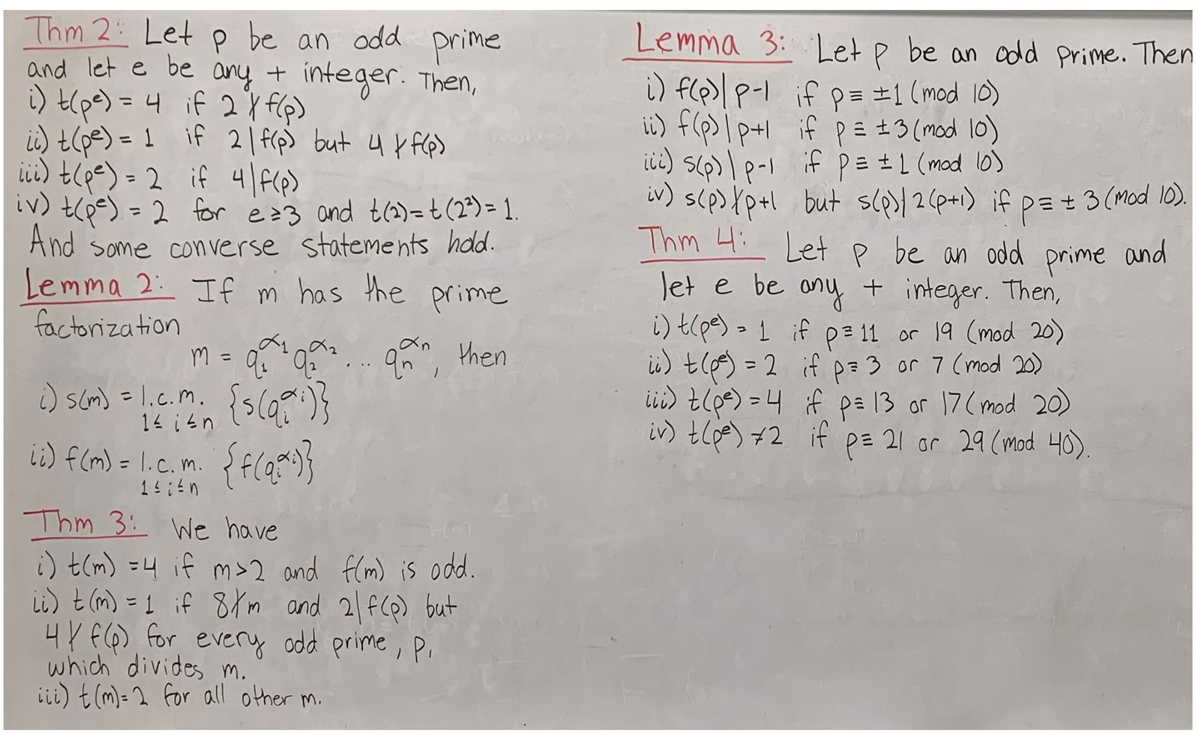
Def 3: We define a function $t(m)$ by the equation $f(m)\,t(m)=s(m)$. We note that $t(m)$ is an integer for all m.

Lemma 1: $t(m)$ is the exponent to which $F_{f(m)-1}$ belongs $\pmod m$.

Thm 1: For $m>2$ we have
i) $t(m)=1$ or 2 if $f(m)$ is even
ii) $t(m)=4$ if $f(m)$ is odd.
Also, $t(1)=t(2)=1$. Conversely, $t(m)=4 \Longrightarrow f(m)$ is odd, $t(m)=2 \Longrightarrow f(m)$ is even, and $t(m)=1 \Longrightarrow f(m)$ is even or $m=1$ or 2.



We remark that the image above gives concrete examples of what Vinson means by an exponent "belonging to" a particular Fibonacci number modulo m. In this example, we have $m = 10$, and we examine all 15 values F_n modulo 10 less than or equal to the rank of apparition of $(\mathcal{F}_{10,n})_{n=0}^{\infty}$ and the numbers in green "belonging to" each value.



Thm 2: Let p be an odd prime and let e be any + integer. Then,
i) $t(p^e) = 4$ if $2 \nmid f(p)$
ii) $t(p^e) = 1$ if $2 \mid f(p)$ but $4 \nmid f(p)$
iii) $t(p^e) = 2$ if $4 \mid f(p)$
iv) $t(p^e) = 2$ for $e \geq 3$ and $t(2) = t(2^2) = 1$.
And some converse statements hold.

Lemma 2: If m has the prime factorization
$$m = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_n^{\alpha_n}, \text{ then}$$
i) $s(m) = \underset{1 \leq i \leq n}{\text{l.c.m.}} \{s(q_i^{\alpha_i})\}$
ii) $f(m) = \underset{1 \leq i \leq n}{\text{l.c.m.}} \{f(q_i^{\alpha_i})\}$

Thm 3: We have
i) $t(m) = 4$ if $m > 2$ and $f(m)$ is odd.
ii) $t(m) = 1$ if $8 \nmid m$ and $2 \mid f(p)$ but $4 \nmid f(p)$ for every odd prime, p, which divides m.
iii) $t(m) = 2$ for all other m.

Lemma 3: Let p be an odd prime. Then
i) $f(p) \mid p-1$ if $p \equiv \pm 1 \pmod{10}$
ii) $f(p) \mid p+1$ if $p \equiv \pm 3 \pmod{10}$
iii) $s(p) \mid p-1$ if $p \equiv \pm 1 \pmod{10}$
iv) $s(p) \nmid p+1$ but $s(p) \mid 2(p+1)$ if $p \equiv \pm 3 \pmod{10}$.

Thm 4: Let p be an odd prime and let e be any + integer. Then,
i) $t(p^e) = 1$ if $p \equiv 11$ or $19 \pmod{20}$
ii) $t(p^e) = 2$ if $p \equiv 3$ or $7 \pmod{20}$
iii) $t(p^e) = 4$ if $p \equiv 13$ or $17 \pmod{20}$
iv) $t(p^e) \neq 2$ if $p \equiv 21$ or $29 \pmod{40}$.

4 Chapter 1 Identities from Vorobiev

Proposition 4.1 (Identity (1.1) [4])**.** *For all $n \geq 1$, we have*

$$F_1 + F_2 + \cdots + F_n = F_{n+2} - 1.$$

Proof. We induct on n.

Base Case: ($n = 1$) Notice the left hand side of our statement as the following: F_1. Notice the right hand side of our statement as the following: $F_3 - 1$. Observe that $F_1 = F_3 - 1 = 1$. Thus our base case is proven. ✓

Induction Hypothesis: Assume $F_1 + F_2 + \cdots + F_{k+1} = F_{k+3} - 1$ for some $k \geq 1$.

WWTS: $F_1 + F_2 + \cdots + F_{k+1} = F_{k+3} - 1.$

Observe the following:

$$\begin{aligned}
F_1 + F_2 + \cdots + F_k + F_{k+1} &= (F_1 + F_2 + \cdots + F_k) + F_{k+1} \\
&= F_{k+2} - 1 + F_{k+1} && \text{by Inductive Hypothesis} \\
&= F_{k+2} + F_{k+1} - 1 \\
&= F_{k+3} - 1 && \text{by Equation (1).}
\end{aligned}$$

Hence, we may conclude $F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$ for all $n \geq 1$. □

Proposition 4.2 (Identity (1.2) [4])**.** *For all $n \geq 1$, we have*

$$F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}.$$

Proof. We induct on n.

Base Case: ($n = 1$) Notice the left hand side of our statement, as the following: $F_{2(1)-1} = F_1$. Notice the right hand side of our statement, as the following: $F_{2(1)} = F_2$. Observe that $F_2 = F_1 = 1$. Thus our base case is proven. ✓

Induction Hypothesis: Suppose $F_1 + F_3 + F_5 + \cdots + F_{2k-1} = F_{2k}$ for some $k \geq 1$.

WWTS: $F_1 + F_3 + F_5 + \cdots + F_{2k+1} = F_{2k+2}.$

Observe the following:

$$\begin{aligned}
F_1 + F_3 + \cdots + F_{2k-1} + F_{2k+1} &= (F_1 + F_3 + \cdots + F_{2k-1}) + F_{2k+1} \\
&= F_{2k} + F_{2k+1} && \text{by Induction Hypothesis} \\
&= F_{2k+2} && \text{by Equation (1).}
\end{aligned}$$

Thus $F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}$ for all $n \geq 1$. □

Proposition 4.3 (Identity (1.3) [4]). *For all $n \geq 1$, we have*

$$F_2 + F_4 + F_6 + \cdots + F_{2n} = F_{2n+1} - 1.$$

Proof. We induct on n.

Base Case: ($n = 1$) Notice the left hand side of our statement as the following: $F_2 = 1$. Notice the right hand side of our statement as the following: $F_3 - 1 = 2 - 1 = 1$. Thus our base case is proven. ✓

Induction Hypothesis: Assume $F_2 + F_4 + F_6 + \cdots + F_{2k} = F_{2k+1} - 1$ for some $k \geq 1$.

WWTS: $F_2 + F_4 + F_6 + \cdots + F_{2k+2} = F_{2k+3} - 1$.

Observe the following:

$$\begin{aligned}
F_2 + F_4 + \cdots + F_{2k} + F_{2k+2} &= (F_2 + F_4 + \cdots + F_{2k}) + F_{2k+2} \\
&= F_{2k+1} - 1 + F_{2k+2} && \text{by Induction Hypothesis} \\
&= F_{2k+1} + F_{2k+2} - 1 \\
&= F_{2k+3} - 1 && \text{by Equation (1).}
\end{aligned}$$

Hence, we may conclude $F_2 + F_4 + F_6 + \cdots + F_{2n} = F_{2n+1} - 1$ for all $n \geq 1$. □

Lemma 4.4. *The following are equivalent to the Fibonacci recurrence $F_n = F_{n-1} + F_{n-2}$:*

$$F_{n-1} = F_n - F_{n-2} \tag{3}$$
$$F_{n-2} = F_n - F_{n-1} \tag{4}$$

Proof. Clear. □

Proposition 4.5 (Identity (1.4) [4]). *For all $n \geq 1$, we have*

$$F_1 - F_2 + F_3 - F_4 + \cdots + F_{2n-2} + F_{2n-1} - F_{2n} = -F_{2n-1} + 1.$$

Proof. Let $n \geq 1$ be given.

WWTS: $F_1 - F_2 + F_3 + \cdots + F_{2n-1} - F_{2n} = -F_{2n-1} + 1.$

Notice the following:

$$\begin{aligned}
&F_1 - F_2 + F_3 - F_4 + \cdots - F_{2n-2} + F_{2n-1} - F_{2n} \\
&\qquad = (F_1 + F_3 + \cdots + F_{2n-1}) - (F_2 + F_4 + \cdots + F_{2n}) \\
&\qquad = F_{2n} - (F_2 + F_4 + \cdots + F_{2n-2} + F_{2n}) && \text{by Proposition 4.2} \\
&\qquad = F_{2n} - (F_{2n+1} - 1) && \text{by Proposition 4.3} \\
&\qquad = F_{2n} - F_{2n+1} + 1 \\
&\qquad = -(F_{2n+1} - F_{2n}) + 1 \\
&\qquad = -F_{2n-1} + 1 && \text{by Equation (4).}
\end{aligned}$$

Thus $F_1 - F_2 + F_3 - F_4 + \cdots + F_{2n-2} + F_{2n-1} - F_{2n} = -F_{2n-1} + 1$ for all $n \geq 1$. □

Proposition 4.6 (Identity (1.5) [4])**.** *For all $n \geq 1$, we have*

$$F_1 + F_2 + F_3 + F_4 + \cdots + F_{2n} + F_{2n+1} = F_{2n} + 1.$$

Proof. Let $n \geq 1$ be given.

WWTS: $F_1 + F_2 + \cdots + F_{2n} + F_{2n+1} = F_{2n} + 1.$

We can write the left side of our statement as the following:

$$\begin{aligned}
&-(F_2 + F_4 + \cdots + F_{2n}) + (F_1 + F_3 + \cdots + F_{2n+1}) \\
&\qquad = (F_1 + F_3 + \cdots + F_{2n+1}) - (F_{2n+1} - 1) && \text{by Proposition 4.3} \\
&\qquad = -F_{2n+1} + F_{2n+2} + 1 && \text{by Proposition 4.2} \\
&\qquad = F_{2n} + 1 && \text{by Equation (4).}
\end{aligned}$$

Hence, we may conclude $F_1 + F_2 + F_3 \cdots + F_{2n} + F_{2n+1} = F_{2n} + 1$ for all $n \geq 1$. □

Proposition 4.7 (Identity (1.6) [4])**.** *For all $m \geq 1$, we have*

$$F_1 - F_2 + F_3 - F_4 + \cdots + (-1)^{m+1} F_m = (-1)^{m+1} F_{m-1} + 1.$$

Proof. Notice there exists two cases. Either m is even or m is odd. We now proceed with the first case.

Case 1: Observe that if m is even, then it can be written as $2n$ for some integer n.

WWTS: $F_1 - F_2 + \cdots + (-1)^{2n+1}F_{2n} = (-1)^{2n+1}F_{2n-1} + 1.$

Notice the following:

$$\begin{aligned}
F_1 - F_2 + F_3 - F_4 \cdots + (-1)^{2n+1}F_{2n} & \\
&= (F_1 + F_3 + \cdots + F_{2n-1}) - (F_2 + F_4 + \cdots + F_{2n}) \\
&= (F_{2n}) - (F_2 + F_4 + \cdots + F_{2n}) && \text{by Proposition 4.2} \\
&= F_{2n} - (F_{2n+1} - 1) && \text{by Proposition 4.3} \\
&= F_{2n} - F_{2n+1} + 1 \\
&= -(F_{2n+1} - F_{2n}) + 1 \\
&= -F_{2n-1} + 1 && \text{by Equation (4)} \\
&= (-1)^{2n+1}F_{2n-1} + 1 \\
&= (-1)^{m+1}F_{m-1} + 1 && \text{by } m = 2n.
\end{aligned}$$

Thus $F_1 - F_2 + F_3 - F_4 + \cdots + (-1)^{m+1}F_m = (-1)^{m+1}F_{m-1} + 1$ whenever m is even. **Case 2:** Let m be odd. Then m can be rewritten as $2n+1$ for some integer n.

WWTS: $F_1 - F_2 + \cdots + (-1)^{2n+2}F_{2n+1} = (-1)^{2n+2}F_{2n} + 1.$

$$\begin{aligned}
F_1 - F_2 + F_3 - F_4 + \cdots + (-1)^{2n+2}F_{2n+1} & \\
&= F_1 - F_2 + F_3 - F_4 + \cdots + F_{2n+1} \\
&= (F_1 + F_3 + \cdots + F_{2n-1}) - (F_2 + F_4 + \cdots + F_{2n}) + F_{2n+1} \\
&= F_{2n} - (F_2 + F_4 + \cdots + F_{2n}) + F_{2n+1} && \text{by Proposition 4.2} \\
&= F_{2n} - (F_{2n+1} - 1) + F_{2n+1} && \text{by Proposition 4.3} \\
&= F_{2n} - F_{2n+1} + F_{2n+1} + 1 \\
&= F_{2n} + 1 \\
&= (-1)^{2n+1+1}F_{2n} + 1
\end{aligned}$$

$$= (-1)^{m+1}F_m + 1 \qquad \text{by } m = 2n+1.$$

Thus $F_1 - F_2 + F_3 - F_4 + \cdots + (-1)^{m+1}F_m = (-1)^{m+1}F_{m-1} + 1$ whenever m is odd. Hence $F_1 - F_2 + F_3 - F_4 + \cdots + (-1)^{m+1}F_m = (-1)^{m+1}F_{m-1} + 1$. for all $m \geq 1$. $\square$

Proposition 4.8 (Identity (1.7) [4])**.** *For all $m \geq 1$, we have*

$$F_1^2 + F_2^2 + \cdots + F_n^2 = F_nF_{n+1}.$$

Proof. We induct on n.

Base Case: $(n = 1)$ Notice the left hand side of our statement is $F_1^2 = 1^2 = 1$, whereas the right hand side of our statement is $F_1F_2 = 1 \cdot 1 = 1$. Observe that $F_1^2 = F_1F_2$. Thus our base case holds. ✓

Induction Hypothesis: Assume $F_1^2 + F_2^2 + \cdots + F_k^2 = F_kF_{k+1}$ for some $k \geq 1$.

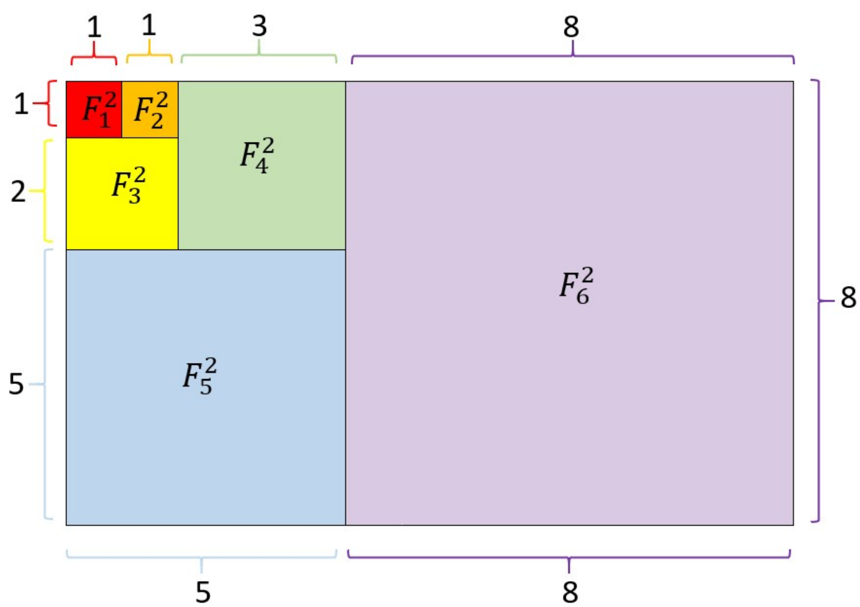
WWTS: $F_1^2 + F_2^2 + \cdots + F_{k+1}^2 = F_{k+1}F_{k+2}$.

Observe the following:

$$\begin{aligned}
F_1^2 + F_2^2 + \cdots + F_{k+1}^2 &= (F_1^2 + F_2^2 + \cdots + F_k) + F_{k+1}^2 \\
&= F_kF_{k+1} + F_{k+1}^2 && \text{by Induction Hypothesis} \\
&= F_{k+1}(F_k + F_{k+1}) \\
&= F_{k+1}F_{k+2} && \text{by Equation (1).}
\end{aligned}$$

Hence, we conclude that $F_1^2 + F_2^2 + \cdots + F_n^2 = F_nF_{n+1}$ holds for all $n \geq 1$. $\square$

Remark 4.9. A geometric "proof without words" is suggested by the following image:



In this diagram, we can clearly see that the sum $F_1^2 + F_2^2 + F_3^2 + F_4^2 + F_5^2 + F_6^2$ equals the area of the rectangle with height $F_6 = 8$ and width given by the sum $F_5 + F_6 = 13$, which is F_7 by the Fibonacci recurrence relation. Hence we have

$$F_1^2 + F_2^2 + F_3^2 + F_4^2 + F_5^2 + F_6^2 = F_6F_7.$$

Proposition 4.10 (Identity (1.8) [4])**.** *For all $m \geq 1$ and n fixed as a natural number, we have*

$$F_{n+m} = F_{n-1}F_m + F_nF_{m+1}.$$

Proof. We induct on m.

Base Cases: $(m = 1)$ **and** $(m = 2)$

$(m = 1)$. Notice the following:

$$F_{n-1}F_1 + F_nF_2 = F_{n-1} + F_n = F_{n+1}.$$

Thus the first base case holds. $\checkmark$

$(m = 2)$. Notice the following:

$$\begin{aligned} F_{n-1}F_2 + F_nF_3 &= F_{n-1} + 2F_n && \text{by } F_2 = 1, F_3 = 2 \\ &= F_n + F_n + F_{n-1} \\ &= F_n + F_{n+1} && \text{by Equation (1)} \\ &= F_{n+2} && \text{by Equation (1).} \end{aligned}$$

Thus the second base case holds. $\checkmark$

Induction Hypotheses: Suppose the following holds for some $k \geq 2$:

$$F_{n+k} = F_{n-1}F_k + F_nF_{k+1}$$

$$F_{n+(k+1)} = F_{n-1}F_{k+1} + F_nF_{k+2}.$$

WWTS: $F_{n+(k+2)} = F_{n-1}F_{k+2} + F_nF_{k+3}$.

Consider the following sequence of equalities:

$$\begin{aligned} &F_{n-1}F_{k+2} + F_nF_{k+3} \\ &\qquad = F_{n-1}(F_k + F_{k+1}) + F_n(F_{k+2} + F_{k+1}) && \text{by Equation (1)} \\ &\qquad = F_{n-1}F_k + F_{n-1}F_{k+1} + F_nF_{k+2} + F_nF_{k+1} \\ &\qquad = F_{n-1}F_k + F_nF_{k+1} + F_{n-1}F_{k+1} + F_nF_{k+2} \\ &\qquad = (F_{n-1}F_k + F_nF_{k+1}) + (F_{n-1}F_{k+1} + F_nF_{k+2}) \end{aligned}$$

$$
\begin{aligned}
&= F_{n+k} + F_{n+k+1} && \text{by Induction Hypotheses} \\
&= F_{n+k+2} && \text{by Equation (1).}
\end{aligned}
$$

Hence $F_{n+m} = F_{n-1}F_m + F_nF_{m+1}$ for all $m \geq 1$. □

Corollary 4.11 (Corollary 1 of Proposition 4.10)**.** *For all $n \geq 1$, we have*

$$F_{2n} = F_{n-1}F_n + F_nF_{n+1}.$$

Proof. Let $n \geq 1$ be given.

WWTS: $F_{2n} = F_{n-1}F_n + F_nF_{n+1}.$

Observe the following relation from Proposition 4.10.

$$F_{n+m} = F_{n-1}F_m + F_nF_{m+1}.$$

Let $m = n$. Then we have the following:

$$F_{n+n} = F_{n-1}F_n + F_nF_{n+1} = F_{2n}.$$

Hence $F_{2n} = F_{n-1}F_n + F_nF_{n+1}$ for all $n \geq 1$. □

Corollary 4.12 (Corollary 2 of Proposition 4.10)**.** *For all $n \geq 1$, we have*

$$F_{2n} = F_{n+1}^2 - F_{n-1}^2.$$

Proof. Let $n \geq 1$ be given.

WWTS: $F_{2n} = F_{n+1}^2 - F_{n-1}^2.$

Observe the following relation from Proposition 4.11:

$$F_{2n} = F_{n-1}F_n + F_nF_{n+1}.$$

Then we have the following:

$$
\begin{aligned}
F_{2n} &= F_n(F_{n-1} + F_{n+1}) \\
&= (F_{n+1} - F_{n-1})(F_{n-1} + F_{n+1}) && \text{by Equation (3)} \\
&= F_{n+1}^2 - F_{n-1}^2.
\end{aligned}
$$

Hence $F_{2n} = F_{n+1}^2 - F_{n-1}^2$ for all $n \geq 1$. □

Remark 4.13. Observe that in the proof of Corollary 4.12, we have another equivalent expression for F_{2n}, namely,

$$F_{2n} = F_n(F_{n-1} + F_{n+1}).$$

Corollary 4.14 (Corollary 3 of Proposition 4.10)**.** *For all $n \geq 1$, we have*

$$F_{2n-1} = F_{n-1}^2 + F_n^2.$$

Proof. Let $n \geq 1$ be given.

WWTS: $F_{2n-1} = F_{n-1}^2 + F_n^2.$

Observe the following sequence of equalities:

$$\begin{aligned}
F_{2n-1} &= F_{n+(n-1)} \\
&= F_{n-1}F_{n-1} + F_nF_n && \text{by Proposition 4.10} \\
&= F_{n-1}^2 + F_n^2.
\end{aligned}$$

Hence $F_{2n-1} = F_{n-1}^2 + F_n^2$ for all $n \geq 1$. ☐

Corollary 4.15 (Corollary 4 of Proposition 4.10)**.** *For all $n \geq 1$, we have*

$$F_{3n} = F_{n+1}^3 + F_n^3 - F_{n-1}^3.$$

Proof. Let $n \geq 1$ be given.

WWTS: $F_{3n} = F_{n+1}^3 + F_n^3 - F_{n-1}^3.$

Observe the folowing sequence of equalities:

$$\begin{aligned}
F_{3n} &= F_{2n} + F_n \\
&= F_{2n-1}F_n + F_{2n}F_{n+1} && \text{by Proposition 4.10} \\
&= (F_n^2 + F_{n-1}^2) \cdot F_n + F_{2n}F_{n+1} && \text{by Corollary 4.14} \\
&= (F_n^2 + F_{n-1}^2) \cdot F_n + (F_{n+1}^2 - F_{n-1}^2) \cdot F_{n+1} && \text{by Corollary 4.12} \\
&= F_n^3 + F_{n-1}^2F_n - F_{n-1}^2F_{n+1} + F_{n+1}^3 \\
&= F_{n+1}^3 + F_n^3 + (F_{n-1}^2F_n - F_{n-1}^2F_{n+1}).
\end{aligned}$$

It suffices to show that $F_{n-1}^2F_n - F_{n-1}^2F_{n+1}$ equals $-F_{n-1}^3$. Observe that we have the following sequence of equalities:

$$F_{n-1}^2F_n - F_{n-1}^2F_{n+1} = F_{n-1}^2(F_n - F_{n-1})$$

$$
\begin{aligned}
&= F_{n-1}^2 \cdot (-F_{n-1}) \\
&= -F_{n-1}^3,
\end{aligned}
$$

where the second equality holds since $F_{n+1} = F_n + F_{n-1}$ implies that $F_n - F_{n+1} = -F_{n-1}$. Hence $F_{3n} = F_{n+1}^3 + F_n^3 - F_{n-1}^3$ for all $n \geq 1$. □

The following is a well-known identity by Jean-Dominique Cassini discovered in 1680.

Proposition 4.16 (Cassini's Identity, also Identity (1.10) [4])**.** *For all $n \geq 2$, we have*

$$F_n^2 = F_{n+1}F_{n-1} + (-1)^{n+1}.$$

Proof. We induct on n.

Base Case: ($n = 2$) Notice the following:

$$F_{2+1}F_{2-1} + (-1)^{2+1} = F_3F_1 + (-1)^3 = 2 - 1 = 1 = F_2^2.$$

Thus our base case holds. ✓

Induction Hypothesis: Suppose $F_k^2 = F_{k+1}F_{k-1} + (-1)^{k+1}$ for some $k \geq 2$.

WWTS: $F_{k+1}^2 = F_{k+2}F_k + (-1)^{k+2}$.

Consider the following sequence of equalities:

$$
\begin{aligned}
F_{k+2}F_k + (-1)^{k+2} &= F_kF_{k+2} + (-1)^{k+2} \\
&= F_k(F_{k+1} + F_k) + (-1)^{k+2} && \text{by Equation (1)} \\
&= F_kF_{k+1} + F_k^2 + (-1)^{k+2} \\
&= F_kF_{k+1} + F_{k+1}F_{k-1} + (-1)^{k+1} + (-1)^{k+2} && \text{by Induction Hypothesis} \\
&= F_kF_{k+1} + F_{k+1}F_{k-1} \\
&= F_{k+1}(F_k + F_{k-1}) \\
&= F_{k+1}(F_{k+1}) && \text{by Equation (1)} \\
&= F_{k+1}^2.
\end{aligned}
$$

Hence $F_n^2 = F_{n+1}F_{n-1} + (-1)^{n+1}$ for all $n \geq 2$. □

5 Golden Ratio, Binet's Formula, and Lucas Numbers

Proposition 5.1 (Sharon, just a gal nextdoor)**.** *For all $n \geq 1$, we have*

$$\phi^n = \phi F_n + F_{n-1}.$$

Proof. We induct on n.

Base Case: **(**$n = 1$**)** Notice the left hand side of our statement as the following: ϕ^1. Notice the right hand side of our statement as the following: $F_1\phi + F_0 = \phi$. Thus our base case has been proven. ✓

Induction Hypothesis: Assume $\phi^k = \phi F_k + F_{k-1}$ for some $k \geq 1$.

WWTS: $\phi^{k+1} = \phi F_{k+1} + F_k$.

Consider the following sequence of equalities:

$$\begin{aligned}
\phi^{k+1} &= \phi^k \cdot \phi \\
&= (F_k\phi + F_{k-1}) \cdot \phi && \text{by Induction Hypothesis} \\
&= \phi^2 F_k + \phi F_{k-1} \\
&= (\phi + 1)F_k + \phi F_{k-1} && \text{since } \phi^2 = \phi + 1 \\
&= \phi F_k + F_k + \phi F_{k-1} \\
&= \phi F_k + \phi F_{k-1} + F_k \\
&= \phi(F_k + F_{k-1}) + F_k \\
&= \phi F_{k+1} + F_k && \text{by Equation (1).}
\end{aligned}$$

Hence, we may conclude $\phi^n = F_n\phi + F_{n-1}$ for all $n \geq 1$. □

Proposition 5.2. *The n^{th} Fibonacci number can be calculated by the following:*

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

Proof. Let n be an integer.

WWTS: $F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$.

Consider the following polynomial:

$$x^2 - x - 1.$$

Notice that this polynomial has two roots, namely α and β. We define these values below:

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2}.$$

Since these two values are roots of the polynomial, it follows that $\alpha^2 = \alpha + 1$ and that $\beta^2 = \beta + 1$. Multiplying the first equation by α^n and the second by β^n, results in the following:

$$\alpha^{n+2} = \alpha^{n+1} + \alpha^n \quad \text{and} \quad \beta^{n+2} = \beta^{n+1} + \beta^n.$$

Subtracting these two equations it follows that

$$\alpha^{n+2} - \beta^{n+2} = \alpha^{n+1} - \beta^{n+1} + \alpha^n - \beta^n.$$

Next we divide both sides by $\alpha - \beta$. Thus it follows that

$$\frac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta} = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} + \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

Denoting H_n as $\frac{\alpha^n - \beta^n}{\alpha^1 - \beta^1}$, it follows that

$$H_{n+2} = H_{n+1} + H_n.$$

Notice this gives the recurrence relation that the Fibonacci sequence follows. To show that $H_1, H_2, H_3, \ldots$ is precisely the Fibonacci sequence, it suffices to show that $H_1 = F_1$ and $H_2 = F_2$. Notice the following:

$$H_1 = \frac{\alpha - \beta}{\alpha - \beta} = 1 = F_1$$

$$\begin{aligned} H_2 = \frac{\alpha^2 - \beta^2}{\alpha - \beta} &= \frac{(\alpha + \beta)(\alpha - \beta)}{\alpha - \beta} \\ &= \alpha + \beta \\ &= \frac{1 + \sqrt{5}}{2} + \frac{1 - \sqrt{5}}{2} \\ &= \frac{1 + \sqrt{5} + 1 - \sqrt{5}}{2} \\ &= 1 \\ &= F_2. \end{aligned}$$

Now that we have shown that H_n follows the Fibonacci recurrence relation with the initial conditions of the sequence. Thus $H_n = F_n$. However, now we must show that H_n is equal to Binet's Formula. Below we illustrate this connection:

$$\begin{aligned} H_n &= \frac{\alpha^n - \beta^n}{\alpha - \beta} \\ &= \frac{\alpha^n - \beta^n}{\sqrt{5}} && \text{Since } \alpha - \beta = \sqrt{5} \\ &= \frac{1}{\sqrt{5}}(\alpha^n - \beta^n) \end{aligned}$$

$$= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n\left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

Therefore $F_n = H_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$ for all integer values of n. $\square$

Definition 5.3. The *Lucas sequence* $(L_n)_{n=0}^{\infty}$ is defined by the recurrence relation

$$L_n = L_{n-1} + L_{n-2} \tag{5}$$

with initial conditions $L_0 = 2$ and $L_1 = 1$. The following table gives the first sixteen Lucas numbers:

$\boldsymbol{n}$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\boldsymbol{L_n}$	2	1	3	4	7	11	18	29	47	76	123	199	322	521	843	1364

Lemma 5.4. *The following are equivalent to the Lucas recurrence* $L_n = L_{n-1} + L_{n-2}$*:*

$$L_{n-1} = L_n - L_{n-2} \tag{6}$$

$$L_{n-2} = L_n - L_{n-1} \tag{7}$$

Proof. Clear. $\square$

Proposition 5.5. *The Lucas sequence has the following closed form*

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Compactly, we write $L_n = \alpha^n + \beta^n$ *where* $\alpha = \frac{1+\sqrt{5}}{2}$ *and* $\beta = \frac{1-\sqrt{5}}{2}$.

Proof. Let n be an integer.

WWTS: $L_n = \alpha^n + \beta^n$.

Consider the following polynomial:

$$x^2 - x - 1.$$

Notice that this polynomial has two roots, namely α and β. We define these values below:

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2}.$$

Since these two values are roots of the polynomial, it follows that $\alpha^2 = \alpha + 1$ and that $\beta^2 = \beta + 1$. Multiplying the first equation by α^n and the second by β^n, results in the following:

$$\alpha^{n+2} = \alpha^{n+1} + \alpha^n \quad \text{and} \quad \beta^{n+2} = \beta^{n+1} + \beta^n.$$

Summing these two equations it follows that

$$\alpha^{n+2} + \beta^{n+2} = \alpha^{n+1} + \beta^{n+1} + \alpha^n + \beta^n.$$

Denoting M_n as $\alpha^n + \beta^n$, it follows that

$$M_{n+2} = M_{n+1} + M_n.$$

Notice this gives the recurrence relation that the Fibonacci sequence follows. To show that $M_1, M_2, M_3, \ldots$ is precisely the Lucas numbers, it suffices to show that $M_1 = L_1$ and $M_2 = L_2$. Consider the following sequence of equalities:

$$\begin{aligned} M_1 &= \alpha + \beta \\ &= 1 \\ &= L_1. \end{aligned}$$

Again, consider the next sequence of equalities:

$$\begin{aligned} M_2 &= \alpha^2 + \beta^2 \\ &= \alpha + 1 + \beta + 1 && \text{since } \alpha \text{ and } \beta \text{ are roots of } x^2 - x - 1. \\ &= 3 && \text{since } \alpha + \beta = 1 = L_2. \end{aligned}$$

Because we have shown that M_n follows the same recurrence relation as the Lucas numbers and has the same two initial values, it follows that $M_n = \alpha^n + \beta^n = L_n$. □

5.1 Proofs of Identities using the Binet-formula technique

In this subsection, we prove some Fibonacci identities using the Binet-formula technique. The equations in the following lemma will be useful in these proofs.

Lemma 5.6. *Let α and β be defined as follows*

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \textit{and} \quad \beta = \frac{1-\sqrt{5}}{2}.$$

Then the following equalities hold:

$$\alpha + \beta = 1 \tag{8}$$

$$\alpha - \beta = \sqrt{5} \tag{9}$$

$$\alpha\beta = -1 \tag{10}$$

$$\alpha^2 = \alpha + 1 \tag{11}$$

$$\beta^2 = \beta + 1 \tag{12}$$

Proof. The validity of Equations (8), (9), and (10) are clear from the definitions of α and β. The validity of Equations (11) and (12) are clear from the fact that α and β are roots of the quadratic $x^2 - x - 1$. □

Proposition 5.7. *For all $n \geq 1$, we have $F_{n+2}^2 - F_n^2 = F_{2n+2}$.*

Proof. Let $n \geq 1$ be given.

WWTS: $F_{n+2}^2 - F_n^2 = F_{2n+2}$.

Consider α and β as given in the Binet formula proof of Proposition 5.2. Observe the following sequence of equalities:

$$
\begin{aligned}
F_{n+2}^2 - F_n^2 &= \left(\frac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta}\right)^2 - \left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)^2 \\
&= \frac{\alpha^{2(n+2)} - 2\alpha^{n+2}\beta^{n+2} + \beta^{2(n+2)}}{(\alpha-\beta)^2} - \frac{\alpha^{2n} - 2\alpha^n\beta^n + \beta^{2n}}{(\alpha-\beta)^2} \\
&= \frac{\alpha^{2(n+2)} - 2\alpha^{n+2}\beta^{n+2} + \beta^{2(n+2)} - \alpha^{2n} + 2\alpha^n\beta^n + \beta^{2n}}{(\alpha-\beta)^2} \\
&= \frac{\alpha^{2(n+2)} - 2(-1)^{n+2} + \beta^{2(n+2)} - \alpha^{2n} + 2(-1)^n + \beta^{2n}}{(\alpha-\beta)^2} && \text{since } \alpha\beta = -1 \\
&= \frac{\alpha^{2(n+2)} + \beta^{2(n+2)} - \alpha^{2n} - \beta^{2n}}{(\alpha-\beta)^2} && \text{since } -2(-1)^{n+2} + 2(-1)^n = 0 \\
&= \frac{\alpha^{2(n+2)} - (\alpha\beta)^2\alpha^{2n} - (\alpha\beta)^2\beta^{2n} + \beta^{2(n+2)}}{(\alpha-\beta)^2} \\
&= \frac{(\alpha^2 - \beta^2)(\alpha^{2n+2} - \beta^{2n+2})}{(\alpha-\beta)^2} \\
&= (\alpha+\beta)\left(\frac{\alpha^{2(n+2)} + \beta^{2(n+2)}}{\alpha - \beta}\right) \\
&= 1 \cdot F_{2n+2} \\
&= F_{2n+2}.
\end{aligned}
$$

Hence we may conclude $F_{n+2}^2 - F_n^2 = F_{2n+2}$ for all $n \geq 1$.

□

Proposition 5.8. *For all $n \geq 1$, we have $F_{2n+1}F_{2n-1} - 1 = F_{2n}^2$.*

Proof. Let $n \geq 1$ be given.

WWTS: $F_{2n+1}F_{2n-1} - 1 = F_{2n}^2$.

Consider α and β as given in the Binet formula proof of Proposition 5.2. Observe the following sequence of equalities:

$$\begin{aligned}
F_{2n+1}F_{2n-1} - 1 &= \frac{\alpha^{2n+1} - \beta^{2n+1}}{\alpha - \beta} \cdot \frac{\alpha^{2n-1} - \beta^{2n-1}}{\alpha - \beta} - 1 \\
&= \frac{\alpha^{4n} + \beta^{4n} - (\alpha\beta)^{2n-1} \cdot \beta^2 - (\alpha\beta)^{2n-1} \cdot \alpha^2}{(\alpha - \beta)^2} - 1 \\
&= \frac{1}{5}\left(\alpha^{4n} + \beta^{4n} - (\alpha\beta)^{2n-1} \cdot (\alpha^2 + \beta^2) - 5\right) && \text{since } \alpha - \beta = \sqrt{5} \\
&= \frac{1}{5}\left(\alpha^{4n} + \beta^{4n} - (-1)^{2n-1} \cdot (\alpha^2 + \beta^2) - 5\right) && \text{since } \alpha\beta = -1 \\
&= \frac{1}{5}\left(\alpha^{4n} + \beta^{4n} + (\alpha^2 + \beta^2) - 5\right) && \text{since } (-1)^{2n-1} = -1 \\
&= \frac{1}{5}\left(\alpha^{4n} + \beta^{4n} + 3 - 5\right) && \text{since } L_2 = \alpha^2 + \beta^2 = 3 \\
&= \frac{1}{5}\left(\alpha^{4n} + \beta^{4n} - 2\right) \\
&= \frac{1}{5}\left(\alpha^{4n} - 2(\alpha\beta)^{2n} + \beta^{4n}\right) && \text{since } \alpha\beta = -1 \\
&= \frac{1}{5}\left(\alpha^{2n} - \beta^{2n}\right) \cdot \left(\alpha^{2n} - \beta^{2n}\right) \\
&= \frac{\alpha^{2n} - \beta^{2n}}{\sqrt{5}} \\
&= F_{2n}.
\end{aligned}$$

Hence $F_{2n+1}F_{2n-1} - 1 = F_{2n}^2$ holds for all $n \geq 1$. $\square$

6 GCD, Divisibility Lemmas, & Congruence Theorems

In this section, we give some well-known GCD and divisibility properties that are used in future sections of the paper.

Lemma 6.1. *If* $b \mid c$*, then* $\gcd(a + c, b) = \gcd(a, b)$.

Proof. Assume b divides c.

WWTS: $\gcd(a + c, b) = \gcd(a, b)$.

Let $d_1 = \gcd(a + c, b)$ and $d_2 = \gcd(a, b)$. Since $d_1, d_2 > 0$, then it suffices to show $d_1 \mid d_2$ and $d_2 \mid d_1$.

Claim 1: $(d_1 \mid d_2)$. Since $d_1 = \gcd(a+c, b)$ then $d_1 \mid (a+c)$ and $d_1 \mid b$. And since $d_1 \mid b$ and $b \mid c$, then $d_1 \mid c$. Hence $d_1 \mid (a+c) - c$. That is, $d_1 \mid a$. So $d_1 \mid a$ and $d_1 \mid b$ implies $d_1 \mid \gcd(a, b)$. Thus $d_1 \mid d_2$.

Claim 2: $(d_2 \mid d_1)$. Since $d_2 = \gcd(a, b)$ then $d_2 \mid a$ and $d_2 \mid b$. And since $d_2 \mid b$ and $b \mid c$, then $d_2 \mid c$. Hence $d_2 \mid (a+c)$. So $d_2 \mid a+c$ and $d_2 \mid b$ implies $d_2 \mid \gcd(a+c, b)$. Thus $d_2 \mid d_1$.

Therefore, $d_1 = d_2$ □

Lemma 6.2. *If* $\gcd(a, c) = 1$, *then* $\gcd(a, bc) = \gcd(a, b)$.

Proof. Assume $\gcd(a, c) = 1$.

WWTS: $\gcd(a, bc) = \gcd(a, b)$.

Notice that there exists two cases. Either $\gcd(a, b) = 1$ or $\gcd(a, b) > 1$. If $\gcd(a, b) = 1$ and $\gcd(a, c) = 1$, then a shares no prime factors with bc. Hence $\gcd(a, bc) = 1 = \gcd(a, b)$. If $\gcd(a, b) > 1$, then a and b have at least one prime factor in common. Hence bc will also contain the prime factor(s). Since $\gcd(a, c) = 1$, we know that a and c contain zero prime factors. Hence the only the common factors between a and bc will come from b, within the product bc. Thus the $\gcd(a, bc) = \gcd(a, b)$. □

Lemma 6.3 (Bézout's Identity)**.** *If* $\gcd(a, b) = 1$, *then there exists* $x, y \in \mathbb{Z}$ *such that* $1 = ax + by$.

Proof. See any number theory textbook. □

Lemma 6.4. *If* $a \mid n$ *and* $b \mid n$ *with* $\gcd(a, b) = 1$, *then* $ab \mid n$.

Proof. Suppose that $a \mid n$ and $b \mid n$ with $\gcd(a, b) = 1$.

WWTS: $ab \mid n$.

Observe the following:

$$\begin{aligned} a \mid n &\implies n = ar \text{ for some } r \in \mathbb{Z} \\ b \mid n &\implies n = bs \text{ for some } s \in \mathbb{Z}. \end{aligned}$$

By Bézout's identity (see Lemma 6.3) we know that since $\gcd(m, n) = 1$, then there exists $x, y \in \mathbb{Z}$ such that $1 = ax + by$. Multiplying across by n, we get $n = axn + byn$. Substituting $n = ar$ and $n = bs$ in the latter we get

$$n = axn + byn = ax(bs) + by(ar) = ab(xs) + by(ar) = ab(xs + ar).$$

Note that $xs + ar$ is clearly an integer and hence $ab \mid n$ as desired. □

Lemma 6.5. *If $a \mid b + c$ and $a \mid b$, then $a \mid c$.*

Proof. Suppose that $a \mid b + c$ and $a \mid b$.

WWTS: $a \mid c$.

Since $a \mid b + c$ and $a \mid b$, then there exists $k, j \in \mathbb{Z}$ such that $b + c = ak$ and $b = aj$. Thus by substitution, we have $b + c = ak$ implies $aj + c = ak$. Thus $c = ak - aj = a(k - j)$. Since $k - j$ is clearly an integer, then we conclude that $a \mid c$ as desired. $\square$

Lemma 6.6. *If $a \mid bc$ and $\gcd(a, b) = 1$, then $a \mid c$.*

Proof. Suppose that $a \mid bc$ and $\gcd(a, b) = 1$.

WWTS: $a \mid c$.

Since $a \mid bc$, then there exists $k \in \mathbb{Z}$ such that $bc = ak$. Moreover since $\gcd(a, b) = 1$, then by Bézout's identity (see Lemma 6.3) there exists $x, y \in \mathbb{Z}$ such that $1 = ax + by$. Multiplying across by c, we get $c = acx + bcy$. And since $bc = ak$, by substitution we have $c = acx + aky$ and hence $c = a(cx + ky)$. Since $cx + ky$ is clearly an integer, then we conclude that $a \mid c$ as desired. $\square$

Theorem 6.7 (Wilson's Theorem)**.** *If p is a prime, then $(p - 1)! \equiv -1 \pmod{p}$.*

Proof. See any number theory textbook. $\square$

7 Fibonacci Identities from Burton

7.1 Proofs discussed by Burton

Proposition 7.1. *For all $n \geq 1$, the following identity holds:*

$$\gcd(F_n, F_{n+1}) = 1.$$

Proof. Let $n \geq 1$ be given. Suppose by way of contradiction that $\gcd(F_n, F_{n+1}) = k$ for some $k > 1$. Therefore, $k \mid F_n$ and $k \mid F_{n+1}$. Hence $k \mid (F_{n+1} - F_n)$. Then, from Equation 4, it follows that $k \mid F_{n-1}$. Since $k \mid F_n$ and $k \mid F_{n-1}$, it follows that $k \mid (F_n - F_{n-1})$. Again, by Equation 4, it follows that $k \mid F_{n-2}$. Repeating this argument, it follows that $k \mid F_1$. Hence $k \mid 1$. However, this is a contradiction because $k > 1$ by assumption. Therefore, $\gcd(F_n, F_{n+1}) = 1$ for all $n \geq 1$. $\square$

Proposition 7.2. *For $m, n \geq 1$, we have F_{mn} is divisible by F_m.*

Proof. We induct on n.

Base Case: $(n = 1)$ $F_m \mid F_{m\cdot 1}$. Thus our base case is proven. ✓

Induction Hypothesis: Assume $F_m \mid F_{mk}$ for some $k \geq 1$. Then $F_{mk} = hF_m$ for some $h \in \mathbb{Z}$.

WWTS: $F_m \mid F_{m\cdot(k+1)}$.

$$
\begin{aligned}
F_{m(k+1)} &= F_{mk+m} \\
&= F_{mk-1}F_m + F_{mk}F_{m+1} && \text{by Proposition 4.10} \\
&= F_{mk-1}F_m + hF_m \cdot F_{m+1} && \text{by Induction Hypothesis} \\
&= F_m(F_{mk-1} + hF_{m+1}) \\
&= F_m \cdot J && \text{where } J = F_{mk-1} + h \cdot F_{m+1} \in \mathbb{Z}.
\end{aligned}
$$

So $F_m \mid F_{m\cdot(k+1)}$. Hence, we may conclude $F_m \mid F_{mn}$ for $m, n \geq 1$. □

Lemma 7.3. *If* $m = qn + r$, *then* $\gcd(F_m, F_n) = \gcd(F_r, F_n)$.

Proof. Let $m, n, q, r \in \mathbb{Z}$ such that $m = qn + r$.

WWTS: $\gcd(F_m, F_n) = \gcd(F_r, F_n)$.

Notice the following:

$$
\begin{aligned}
\gcd(F_m, F_n) &= \gcd(F_{qn+r}, F_n) && \text{by assumption} \\
&= \gcd\left(F_{qn-r}F_r + F_{qn}F_{r+1}, F_n\right) && \text{by Proposition 4.10.}
\end{aligned}
$$

Observe the fact that $F_n \mid F_{qn}$, via Proposition 7.2. Thus, it follows that $F_n \mid F_{qn}F_{r+1}$. Therefore, via Lemma 6.1, we have the following:

$$\gcd(F_{qn-r}F_r + F_{qn}F_{r+1}, F_n) = \gcd(F_{qn-r}F_r, F_n).$$

Now let $d = \gcd(F_{qn-1}, F_n)$, then $d \mid F_n$ and $F_n \mid F_{qn}$, by Proposition 7.2. Hence, by transitivity, $d \mid F_{qn}$. However, d also divides F_{qn-1}. Thus, by Corollary 7.6, $d = 1$. Hence $\gcd(F_{qn-1}, F_n) = 1$. Therefore, using Proposition 7.2, it can be seen that

$$\gcd(F_{qn-r}F_r, F_n) = \gcd(F_r, F_n).$$

Therefore, we have proved that if $m = qn + r$, then $\gcd(F_m, F_n) = \gcd(F_r, F_n)$. □

Proposition 7.4. *The greatest common divisor of two Fibonacci numbers is again a Fibonacci number. Specifically, we have*

$$\gcd(F_m, F_n) = F_d \text{ where } d = \gcd(m, n).$$

Proof. Let $n, m \geq 1$ be given.

WWTS: $\gcd(F_m, F_n) = F_d$ where $d = \gcd(m, n)$.

We begin our proof by running Euclid's algorithm for a generic m and n. We illustrate this below:

$$\begin{aligned}
m &= q_1 n + r_1 \\
n &= q_2 r_1 + r_2 \\
&\vdots \\
r_{n-2} &= q_n r_{n-1} + r_n \\
r_{n-1} &= q_{n+1} r_n + 0.
\end{aligned}$$

From Lemma 7.3, the following holds:

$$\gcd(F_m, F_n) = \gcd(F_n, F_{r_1}) = \cdots = \gcd(F_{r_{n-1}}, F_{r_n}) = \gcd(F_{r_n}, 0) = F_{r_n}.$$

Hence $\gcd(F_m, F_n) = F_{r_n}$. However, from Euclid's algorithm, it can be seen that r_n equals $\gcd(m, n) = d$. Hence it follows that $\gcd(F_m, F_n) = F_{r_n} = F_d$. □

Corollary 7.5. *For all $n \geq m \geq 3$, we have $F_m \mid F_n$ if and only if $m \mid n$.*

Proof. Suppose $F_m \mid F_n$.

WWTS: $m \mid n$.

Then $\gcd(F_m, F_n) = F_m$. Therefore from Proposition 7.4 we conclude the following:

$$F_m = \gcd(F_m, F_n) = F_{\gcd(m,n)}.$$

Since $m, n \geq 3$, the only way for the equation $F_m = F_{\gcd(m,n)}$ to hold is if the subscripts are equal. Thus $m = \gcd(m, n)$. Hence $m \mid n$. Therefore, $m \mid n$, if $F_m \mid F_n$.

Suppose $m \mid n$.

WWTS: $F_m \mid F_n$.

Then $\gcd(m,n) = m$. Therefore from Proposition 7.4, it can be seen that $\gcd(F_m, F_n) = F_m$. Hence $F_m \mid F_n$. Therefore, $F_m \mid F_n$ if $m \mid n$. Thus we can conclude that $F_m \mid F_n$ if and only if $m \mid n$.

□

Corollary 7.6. *For all $n \geq 0$, it follows that F_n and F_{n+1} are relatively prime.*

Proof. Notice that the $\gcd(n, n+1) = 1$ for all $n \geq 0$. Hence from Proposition 7.4 it follows that

$$\gcd(F_n, F_{n+1}) = F_{\gcd(n,n+1)} = F_1 = 1.$$

Hence we may conclude that F_n and F_{n+1} are relatively prime for all $n \geq 0$. □

7.2 Exercise Solutions

Exercises from Section 14.2 from Burton

Claim 7.7 (Exercise 1 from 14.2 [1]). *Given prime $p \neq 5$, then either F_{p-1} or F_{p+1} is divisible by p. Confirm this in the cases of the primes 7, 11, 13, and 17.*

Solution:

$$\begin{aligned}
&F_{7+1} = F_8 = 21 = 7 \cdot 3 \text{ and so } 7 \mid 21 \checkmark \\
&F_{11-1} = F_{10} = 55 = 11 \cdot 5 \text{ and so } 11 \mid 55 \checkmark \\
&F_{13+1} = F_{14} = 377 = 13 \cdot 29 \text{ and so } 13 \mid 377 \checkmark \\
&F_{17+1} = F_{18} = 2584 = 17 \cdot 152 \text{ and so } 17 \mid 2584 \checkmark
\end{aligned}$$

Proposition 7.8 (Exercise 2 from 14.2 [1]). *For all $n = 1, 2, \ldots 10$, show that $F_n^2 + 4(-1)^n$ is always a perfect square.*

Notice the following:

$$\begin{aligned}
5F_1^2 + (-4) = 5(1)^2 + (-4) &= 1 \checkmark \\
5F_2^2 + 4 = 5F(1)^2 + (-4) &= 9 \checkmark \\
5F_3^2 + (-4) = 5(2)^2 + (-4) &= 16 \checkmark \\
5F_4^2 + 4 = 5(3)^2 + (-4) &= 49 \checkmark \\
5F_5^2 + (-4) = 5(5)^2 + (-4) &= 121 \checkmark \\
5F_6^2 + 4 = 5(8)^2 + (-4) &= 324 = 18^2 \checkmark \\
5F_7^2 + (-4) = 5(13)^2 + (-4) &= 841 = 29^2 \checkmark \\
5F_8^2 + 4 = 5(21)^2 + (-4) &= 2209 = 47^2 \checkmark \\
5F_9^2 + (-4) = 5(34)^2 + (-4) &= 5776 = 76^2 \checkmark \\
5F_{10}^2 + 4 = 5(55)^2 + (-4) &= 15129 = 123^2 \checkmark
\end{aligned}$$

Hence $F_n^2 + 4(-1)^n$ is always a perfect square whenever $n = 1, 2, \ldots, 10$.

Proposition 7.9 (Exercise 3 from 14.2 [1])**.** *Prove that if* $2 \mid F_n$*, then* $4 \mid F_{n+1}^2 - F_{n-1}^2$*. And similarly, if* $3 \mid F_n$ *then* $9 \mid F_{n+1}^3 - F_{n-1}^3$*.*

Proof. We prove the first implication. Assume that $2 \mid F_n$. Then $F_n = 2J$ for some $J \in \mathbb{Z}$.

WWTS: $4 \mid F_{n+1}^2 - F_{n-1}^2$.

Observe the following sequence of equalities:

$$\begin{aligned} F_{n+1}^2 - F_{n-1}^2 &= (F_{n+1} - F_{n-1})(F_{n+1} + F_{n-1}) \\ &= F_n \cdot (F_{n+1} + F_{n-1}) && \text{by Equation (3)} \\ &= 2J \cdot 2K && \text{see justification below} \\ &= 4 \cdot JK. \end{aligned}$$

The third equality follows for the following reason. Recall that adjacent Fibonacci numbers are relatively prime which forces F_k and F_{k+1} to never both be even for all k. Since we know that F_n is even, then neither F_{n-1} nor F_{n+1} can be even so they are both odd. Thus $F_{n+1} + F_{n-1}$ is necessarily even and so $F_{n+1} + F_{n-1} = 2K$ for some $K \in \mathbb{Z}$. Since we have $F_{n+1}^2 - F_{n-1}^2 = 4 \cdot JK$, then we conclude that $4 \mid F_{n+1}^2 - F_{n-1}^2$ as desired.

We now prove the second implication. Assume that $3 \mid F_n$. Then $F_n = 3J$ for some $J \in \mathbb{Z}$.

WWTS: $9 \mid F_{n+1}^3 - F_{n-1}^3$.

Observe the following sequence of equalities:

$$\begin{aligned} F_{n+1}^3 - F_{n-1}^3 &= F_{3n} - F_n^3 && \text{by Corollary 4.15} \\ &= F_{3n} - (3J)^3 && \text{since } F_n = 3J \text{ by assumption} \\ &= F_{3n} - 9 \cdot (3J^3). \end{aligned}$$

Hence it suffices to show that 9 divides F_{3n}. Since $F_4 = 3$ and $3 \mid F_n$, we have that $F_4 \mid F_n$. But by Corollary 7.5, we have $F_4 \mid F_n$ implies $4 \mid n$. Thus $n = 4k$ for some $k \in \mathbb{Z}$. Hence $F_{3n} = F_{3 \cdot 4k} = F_{12k}$ and thus $F_{12} \mid F_{12k}$ by Lemma 7.2 (that is, $F_{12} \mid F_{3n}$). But $F_{12} = 144 = 9 \cdot 16$ and hence $9 \mid F_{12}$. Thus we have

$$9 \mid F_{12} \text{ and } F_{12} \mid F_{3n} \quad \Longrightarrow \quad 9 \mid F_{3n}.$$

So we know $F_{3n} = 9K$ for some $K \in \mathbb{Z}$. Since $F_{n+1}^3 - F_{n-1}^3 = F_{3n} - 9 \cdot (3J^3)$ from the sequence of equalities above, we may write $F_{n+1}^3 - F_{n-1}^3 = 9K - 9 \cdot (3J^3) = 9(K - 3J^3)$ and hence 9 divides $F_{n+1}^3 - F_{n-1}^3$ as desired. $\square$

Proposition 7.10 (Exercise 4 from 14.2 [1])**.** *Prove the following:*

(a) $F_{n+3} \equiv F_n \pmod 2$*, and*

(b) $F_{n+5} \equiv 3F_n \pmod 5$*.*

Proof. (for Part(a)) Let $n \geq 1$ be given.

WWTS: $F_{n+3} \equiv F_n \pmod 2$.

It suffices to show that the difference $F_{n+3} \equiv F_n$ is divisible by 2. To this end, observe the following:

$$\begin{aligned} F_{n+3} &= F_{3+n} \\ &= F_2F_n + F_3F_{n+1} && \text{by Proposition 4.10} \\ &= 1F_n + 2F_{n+1} && \text{since } F_2 = 1 \text{ and } F_3 = 2. \end{aligned}$$

Thus we have $F_{n+3} - F_n = 2K$ where $K = F_{n+1} \in \mathbb{Z}$. Hence $F_{n+3} \equiv F_n \pmod 5$ as desired. □

Proof. (for Part (b)) Let $n \geq 1$ be given.

WWTS: $F_{n+5} \equiv 3F_n \pmod 5$.

It suffices to show that the difference $F_{n+5} - 3F_n$ is divisible by 5. To this end, observe the following:

$$\begin{aligned} F_{n+5} &= F_{5+n} \\ &= F_4F_n + F_5F_{n+1} && \text{by Proposition 4.10} \\ &= 3F_n + 5F_{n+1} && \text{since } F_4 = 3 \text{ and } F_5 = 5. \end{aligned}$$

Thus we have $F_{n+5} - 3F_n = 5K$ where $K = F_{n+1} \in \mathbb{Z}$. Hence $F_{n+5} \equiv 3F_n \pmod 5$ as desired. □

Proposition 7.11 (Exercise 5 from 14.2 [1])**.** *Prove that the sum of the squares of the first n Fibonacci numbers is given by the formula*

$$F_1^2 + F_2^2 + \cdots + F_n^2 = F_nF_{n+1}.$$

Proof. See Proposition 4.8. □

Lemma 7.12. *For all $k \geq 1$, we have the following:*

$$F_k^2 = F_kF_{k+1} - F_{k-1}F_k.$$

Proof. Observe the following sequence of equalities:

$$
\begin{aligned}
F_k^2 &= F_k F_k \\
&= F_k(F_{k+1} - F_{k-1}) && \text{by Equation (3)} \\
&= F_k F_{k+1} - F_{k-1} F_k.
\end{aligned}
$$

Hence the claim holds. □

Proposition 7.13 (Exercise 6 from 14.2 [1]). *Prove that for all $n \geq 3$, we have*

$$F_{n+1}^2 = F_n^2 + 3F_{n-1}^2 + 2\big(F_{n-2}^2 + F_{n-3}^2 + \cdots + F_2^2 + F_1^2\big).$$

Proof. Let $n \geq 3$ be given.

WWTS: $F_{n+1}^2 = F_n^2 + 3F_{n-1}^2 + 2\big(F_{n-2}^2 + F_{n-3}^2 + \cdots + F_1^2\big)$.

By repeated use of Lemma 7.12, which states that $F_k^2 = F_k F_{k+1} - F_{k-1} F_k$ for all $k \geq 1$, we have

$$
\begin{aligned}
&F_n^2 + 3F_{n-1}^2 + 2\big(F_{n-2}^2 + F_{n-3}^2 + \cdots + F_2^2 + F_1^2\big) \\
&\qquad = (F_n F_{n+1} - F_{n-1} F_n) + 3(F_{n-1} F_n - F_{n-2} F_{n-3}) + 2\sum_{i=1}^{n-2} (F_i F_{i+1} - F_{i-1} F_i) \\
&\qquad = (F_n F_{n+1} - F_{n-1} F_n) + 3(F_{n-1} F_n - F_{n-2} F_{n-3}) + 2(-F_0 F_1 + F_{n-2} F_{n-1}) \\
&\qquad = F_n F_{n+1} + 2F_{n-1} F_n - F_{n-2} F_{n-1},
\end{aligned}
$$

where this second equality holds since the sum in the first equality is a telescoping sum, and the third equality holds by observing that $F_0 = 0$ and collecting like-terms. Noting that F_{n+1}^2 equals $F_{n+1} F_{n+2} - F_n F_{n+1}$ by Lemma 7.12, it suffices to show that

$$F_n F_{n+1} + 2F_{n-1} F_n - F_{n-2} F_{n-1} = F_{n+1} F_{n+2} - F_n F_{n+1}.$$

Observe the following sequence of equalities:

$$
\begin{aligned}
F_n F_{n+1} + 2F_{n-1} F_n - F_{n-2} F_{n-1} &= (F_n F_{n+1} + F_{n-1} F_n) + (F_{n-1} F_n - F_{n-2} F_{n-1}) \\
&= F_n (F_{n+1} + F_{n-1}) + F_{n-1}(F_n - F_{n-2}) \\
&= F_{2n} + F_{n-1}(F_n - F_{n-2}) && \text{by Remark 4.13} \\
&= F_{2n} + F_{n-1} F_{n-1} && \text{by Equation (3)} \\
&= F_{2n} + F_{n-1}^2 \\
&= F_{n+1}^2,
\end{aligned}
$$

where the last equality holds since Corollary 4.12, which states that $F_{2n} = F_{n+1}^2 - F_{n-1}^2$, will imply that $F_{2n} + -F_{n-1}^2 = F_{n+1}^2$. Thus the claim holds. □

Proposition 7.14 (Exercise 7 from 14.2 [1]). *Evaluate* $\gcd(F_9, F_{12}), \gcd(F_{15}, F_{20}), \gcd(F_{24}, F_{36})$.

Proof. Using Proposition 7.4, we know that $\gcd(F_m, F_n) = F_{\gcd(m,n)}$.

$$\begin{aligned}
\gcd(F_9, F_{12}) &= F_{\gcd(9,12)} = F_3 = 2 \\
\gcd(F_{15}, F_{20}) &= F_{\gcd(15,20)} = F_5 = 5 \\
\gcd(F_{24}, F_{36}) &= F_{\gcd(24,36)} = F_6 = 8.
\end{aligned}$$

$\square$

Proposition 7.15 (Exercise 8 from 14.2 [1]). *Find the Fibonacci numbers that divide both F_{24} and F_{36}.*

Proof. From Corollary 7.5, we simply we must find the common divisors of 24 and 36. Hence the common divisors of 24 and 36 are $1, 2, 3, 4, 6, 12$. Therefore, the Fibonacci numbers that divide both F_{24} and F_{36} are $F_1, F_2, F_3, F_4, F_6, F_{12}$. To illustrate this statement further, we write the prime factorizations of $F_1, F_2, F_3, F_4, F_6, F_{12}, F_{24}$ and F_{36} below:

$$\begin{aligned}
F_1 &= 2^0 \\
F_2 &= 2^0 \\
F_3 &= 2^2 \\
F_4 &= 3^1 \\
F_6 &= 2^3 \\
F_{12} &= 2^4 \cdot 3^2 \\
F_{24} &= 2^5 \cdot 3^2 \cdot 7 \cdot 23. \\
F_{36} &= 2^4 \cdot 3^3 \cdot 17 \cdot 19 \cdot 107.
\end{aligned}$$

$\square$

Proposition 7.16 (Exercise 9 from 14.2 [1]). *Use the fact that $F_m \mid F_n$ if and only if $m \mid n$ (i.e., Corollary 7.5) to verify each of the assertions below:*

(a) $2 \mid F_n$ *if and only if* $3 \mid n$.

(b) $3 \mid F_n$ *if and only if* $4 \mid n$.

(c) $5 \mid F_n$ *if and only if* $5 \mid n$.

(d) $8 \mid F_n$ *if and only if* $6 \mid n$.

Proof. Let $n \geq 0$ be given.

Proof of (a):

WWTS: $2 \mid F_n$ if and only if $3 \mid n$.

By the numerical value of $F_3 = 2$ and Corollary 7.5, respectively, we have

$$2 \mid F_n \iff F_3 \mid F_n \iff 3 \mid n.$$

Hence the claim holds.

Proof of (b):

WWTS: $3 \mid F_n$ if and only if $4 \mid n$.

By the numerical value of $F_4 = 3$ and Corollary 7.5, respectively, we have

$$3 \mid F_n \iff F_4 \mid F_n \iff 4 \mid n.$$

Hence the claim holds.

Proof of (c):

WWTS: $5 \mid F_n$ if and only if $5 \mid n$.

By the numerical value of $F_5 = 5$ and Corollary 7.5, respectively, we have

$$5 \mid F_n \iff F_5 \mid F_n \iff 5 \mid n.$$

Hence the claim holds.

Proof of (d):

WWTS: $8 \mid F_n$ if and only if $6 \mid n$.

By the numerical value of $F_6 = 8$ and Corollary 7.5, respectively, we have

$$8 \mid F_n \iff F_6 \mid F_n \iff 6 \mid n.$$

Hence the claim holds. ☐

Proposition 7.17 (Exercise 10 from 14.2 [1]). *If* $\gcd(m, n) = 1$*, prove* $F_m F_n$ *divides* F_{mn} *for all* $m, n \geq 1$*.*

Proof. Let $m, n \geq 1$ be given. Assume $\gcd(m, n) = 1$.

WWTS: $F_m F_n \mid F_{mn}$.

By Proposition 7.4, we know $\gcd(F_m, F_n) = F_d$ where $d = \gcd(m, n)$. From this we get $\gcd(F_m, F_n) = F_1 = 1$. From Corollary 7.5, we know that $F_m \mid F_{mn}$ and $F_n \mid F_{mn}$. And we know the $\gcd(F_m, F_n) = 1$. We can conclude $F_m F_n \mid F_{mn}$ by Lemma 6.4.

□

Proposition 7.18 (Exercise 11 from 14.2 [1])**.** *It can be shown that when F_n is divided by F_m where $n > m$, then the remainder r is a Fibonacci number or $F_m - r$ is a Fibonacci number. Give examples illustrating both cases.*

Proof. Consider the following: $F_{12} = 144 = 28 \cdot 5 + 4 = 28 \cdot F_5 + 4$. Notice $5 - 4 = 1 = F_1$. Here we have illustrated the case of when $F_m - r$ is a Fibonacci number. Below we show two cases of when the remainder is a Fibonacci number.

$$F_{24} = 233 \cdot F_{13} + 1$$

$$F_{23} = 199 \cdot F_{12} + 1$$

Notice 1 is a Fibonacci number. Therefore, our two cases have been illustrated. □

Proposition 7.19 (Exercise 12 from 14.2 [1])**.** *It was proven in 1989 that ther are only five Fibonacci numbers that are also triangular numbers. Find them.*

Proof. By brute force we give the five solutions:

$$\begin{aligned}
F_0 &= 0 = T_0 \\
F_1 &= 1 = T_1 \\
F_4 &= 3 = 1 + 2 = T_2 \\
F_8 &= 21 = 1 + 2 + 3 + 4 + 5 + 6 = T_6 \\
F_{10} &= 55 = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = T_{10}
\end{aligned}$$

□

Remark 7.20. It is known that the proof for Proposition 7.19 above involves the following biconditionals:

$$F_n \text{ is triangular} \iff 8F_n + 1 \text{ is a square} \iff n \in \{\pm 1, 0, 2, 4, 8, 10.$$

Proposition 7.21 (Exercise 13 from 14.2 [1])**.** *Prove* $2^{n-1} F_n \equiv n (\text{mod } 5)$.

Proof. We induct on n.

Base Cases: $(n = 1)$ and $(n = 2)$
$(n = 1)$ Observe that $2^{1-1} F_1 = 1 \equiv 1 \pmod 5$. ✓
$(n = 2)$ Observe that $2^{2-1} F_2 = 2 \equiv 2 \pmod 5$. ✓

Induction Hypotheses: Suppose the following for some $k \geq 2$:

$$2^{k-2}F_{k-1} \equiv k-1 \pmod 5$$
$$2^{k-1}F_k \equiv k \pmod 5$$

WWTS: $2^k F_{k+1} \equiv (k+1) \pmod 5$.

$$\begin{aligned}
2^k F_{k+1} &= (2^{k-1} \cdot 2^1)F_{k+1} && \text{by Properties of Exponents} \\
&= (2^{k-1} \cdot 2)(F_k + F_{k-1}) && \text{by Equation (1)} \\
&= 2^{k-1} \cdot 2 \cdot F_k + 2^{k-1} \cdot 2 \cdot F_{k-1} \\
&= 2 \cdot (2^{k-1} \cdot F_k) + 2 \cdot 2 \cdot (2^{k-2} \cdot F_{k-1}) && \text{by Properties of Exponents} \\
&\equiv 2 \cdot k + 2 \cdot 2 \cdot (k-1) \pmod 5 && \text{by Induction Hypotheses} \\
&\equiv 2k + 4k - 4 \pmod 5 \\
&\equiv 6k - 4 \pmod 5 \\
&\equiv k - 4 \pmod 5 \\
&\equiv k + 1 \pmod 5
\end{aligned}$$

Hence, we may conclude $2^{n-1}F_n \equiv n \pmod 5$ for all $n \geq 1$.

□

Proposition 7.22 (Exercise 14 from 14.2 [1])**.** *If $F_n < a < F_{n+1} < b < F_{n+2}$ for some $n \geq 4$ establish that the sum $a+b$ cannot be a Fibonacci number.*

Proof. Assume $F_n < a < F_{n+1} < b < F_{n+2}$ for some $n \geq 4$.

WWTS: $a + b$ cannot be a Fibonacci number.

Notice that, via the Fibonacci recurrence relation, $F_{n+2} = F_{n+1} + F_n$. Since $a > F_n$ and $b > F_{n+1}$, then it follows that $F_{n+2} < a + b$. Also, $F_{n+3} = F_{n+2} + F_{n+1}$ by the Fibonacci recurrence relation. Since $a < F_{n+1}$ and $b < F_{n+2}$, it follows that $F_{n+3} > a + b$. Thus the sum $a + b$ lies between F_{n+2} and F_{n+3}. However there exists no Fibonacci number between F_{n+2} and F_{n+3}. Thus $a + b$ can not be a Fibonacci number. □

Proposition 7.23 (Exercise 15 from 14.2 [1])**.** *Prove that there is no positive integer n for which $F_1 + F_2 + F_3 + \cdots + F_{3n} = 16!$.*

Proof. Suppose by way of contradiction that $F_1 + F_2 + F_3 + \cdots + F_{3n} = 16!$. By Wilson's theorem (see Theorem 6.7), we have $(17-1)! \equiv -1 \pmod{17}$, and hence $16! \equiv -1 \pmod{17}$. Thus since $F_1 + F_2 + F_3 + \cdots + F_{3n} = 16!$, then it follows that

$$F_1 + F_2 + F_3 + \cdots + F_{3n} \equiv -1 \pmod{17}.$$

But by Proposition 4.1, we know the sum $F_1 + F_2 + F_3 + \cdots + F_{3n}$ equals $F_{3n+2} - 1$. Thus we have $F_{3n+2} - 1 \equiv -1 \pmod{17}$. And hence $F_{3n+2} \equiv 0 \pmod{17}$. But 17 does not divide F_{3n+2} for any n by Theorem 9.5. Hence we have a contradiction. So we conclude that there is no positive integer n for which $F_1 + F_2 + F_3 + \cdots + F_{3n} = 16!$. □

Remark 7.24. In the table below, we give a BRUTE-FORCE method to solve the previous exercise. Observe that 16! = 20,922,789,888,000. From the table below, it is clear that from $n = 21$ to $n = 22$, the value $F_1 + F_2 + F_3 + \cdots + F_{3n}$ just exceeds $16!$. Hence no n-value exists such that $F_1 + F_2 + F_3 + \cdots + F_{3n} = 16!$.

n	$F_1 + F_2 + F_3 + \cdots + F_{3n}$
1	4
2	20
3	88
4	376
5	1,596
6	6,764
7	28,656
8	121,392
9	514,228
10	2,178,308
11	9,227,464
12	39,088,168
13	165,580,140
14	701,408,732
15	2,971,215,072
16	12,586,269,024
17	53,316,291,172
18	225,851,433,716
19	956,722,026,040
20	4,052,739,537,880
21	17,167,680,177,564
22	72,723,460,248,140

Lemma 7.25. *If $n \in 3\mathbb{Z} + 1$ or $n \in 3\mathbb{Z} + 2$, then F_n is odd.*

Proof. Assume that $n \in 3\mathbb{Z} + 1$ or $n \in 3\mathbb{Z} + 2$.

WWTS: F_n is odd.

As a consequence of Corollary 7.6, we know that there will never be two consecutive even Fibonacci numbers. Also since by part (a) of Proposition 7.16, we know if all Fibonacci of the form F_m with $m \in 3\mathbb{Z}$ are even. Hence if $n \in 3\mathbb{Z}+1$ or $n \in 3\mathbb{Z}+2$, then F_n is forced to be odd. $\square$

Proposition 7.26 (Exercise 16 from 14.2 [1])**.** *If* 3 *divides* $n+m$*, show that*

$$F_{n-m-1}F_n + F_{n-m}F_{n+1}$$

is an even integer.

Proof. Assume that 3 divides $n+m$.

WWTS: $2 \mid F_{n-m-1}F_n + F_{n-m}F_{n+1}$.

There are two cases we must show to prove this statement.

Case 1: Assume 3 divides n and 3 divides m. We can state the following:

$$\begin{aligned} 3 \mid n &\Rightarrow F_3 \mid F_n && \text{by Corollary 7.4}\\ &\Rightarrow 2 \mid F_n \\ &\Rightarrow F_n \text{ is even.} \end{aligned}$$

We can say the same for m and F_m. Since $3 \mid n$ and $3 \mid m$, then $3 \mid n-m$. From this we can state the following:

$$\begin{aligned} 3 \mid n-m &\Rightarrow F_3 \mid F_{n-m} && \text{by Corollary 7.4}\\ &\Rightarrow 2 \mid F_{n-m} \\ &\Rightarrow F_{n-m} \text{ is even.} \end{aligned}$$

From our above statements we can write that $F_n = 2K$ for some $K \in \mathbb{Z}$. And $F_{n-m} = 2J$ for some $J \in \mathbb{Z}$. And we can use our original equation to write,

$$\begin{aligned} F_{n-m-1}F_n + F_{n-m}F_{n+1} &= F_{n-m-1}2K + 2JF_{n+1}\\ &= 2(F_{n-m-1}K + JF_{n+1}). \end{aligned}$$

Hence $F_{n-m-1}F_n + F_{n-m}F_{n+1}$ is an even integer.

Case 2: Assume 3 does not divide n and 3 does not divide m. There are four subcases:

- (Subcase 1): $n = 3a + 1$ and $m = 3b + 2$.
- (Subcase 2): $n = 3a + 2$ and $m = 3b + 1$.
- (Subcase 3): $n = 3a + 1$ and $m = 3b + 1$.
- (Subcase 4): $n = 3a + 2$ and $m = 3b + 2$.

(Subcase 1): Let $n = 3a+1$ and $m = 3b+2$ for some $a, b \in \mathbb{Z}$. We can state the following:

$$\begin{aligned} F_{n-m-1}F_n + F_{n-m}F_{n+1} &= F_{(3a+1)-(3b+2)-1}F_{3a+1} + F_{(3a+1)-(3b+2)}F_{(3a+1)+1} \\ &= F_{3(a-b)-2}F_{3a+1} + F_{3(a-b)-1}F_{3a+2} \end{aligned}$$

We know that $F_{3(a-b)-2}, F_{3a+1}$, $F_{3(a-b)-1}$ and F_{3a+2} are all odd from Lemma 7.25. Hence $F_{(3a+1)-(3b+2)-1}F_{3a+1}$ and $F_{(3a+1)-(3b+2)}F_{(3a+1)+1}$ are both odd. We also know that the sum of two odd numbers is an even number. Hence, we may conclude $2 \mid F_{n-m-1}F_n + F_{n-m}F_{n+1}$ when $3 \mid n+m$.

(Subcase 2): Let $n = 3a+2$ and $m = 3b+1$ for some $a, b \in \mathbb{Z}$. We can state the following:

$$\begin{aligned} F_{n-m-1}F_n + F_{n-m}F_{n+1} &= F_{(3a+2)-(3b+1)-1}F_{3a+2} + F_{(3a+2)-(3b+1)}F_{(3a+2)+1} \\ &= F_{3(a-b)}F_{3a+2} + F_{3(a-b)+1}F_{3a+3} \end{aligned}$$

We know that $F_{3(a-b)}$ and F_{3a+3} are both even from part (a) of Proposition 7.16. Hence $F_{3(a-b)}F_{3a+2}$ and $F_{3(a-b)+1}F_{3a+3}$ are both even. We also know that the sum of two even numbers is an even number. Hence, we may conclude $2 \mid F_{n-m-1}F_n + F_{n-m}F_{n+1}$ when $3 \mid n+m$.

(Subcase 3): Let $n = 3a + 1$ and $m = 3b + 1$ for some $a, b \in \mathbb{Z}$. Thus it follows that $n + m = (3a + 1) + (3b + 1) = 3(a + b) + 2$. Hence 3 does not divide $n + m$, which violates our original assumption. Therefore this subcase is frog food.

(Subcase 4): Let $n = 3a + 2$ and $m = 3b + 2$ for some $a, b \in \mathbb{Z}$. Thus it follows that $n+m = (3a+2)+(3b+2) = 3(a+b+1)+1$. Hence 3 does not divide $n+m$, which violates our original assumption. Therefore this subcase is frog food.

$\square$

Proposition 7.27 (Exercise 17 from 14.2 [1])**.** *For all $n \geq 1$, Verify that there exist n consecutive composite Fibonacci numbers.*

Proof. Let $n \geq 1$ be given.

WWTS: $\exists n$ consecutive composite Fibonacci numbers.

From the Corollary 7.5, it follows that whenever $n > 4$, then F_n will be composite. Hence it suffices to show that there exists n consecutive composite numbers for all $n > 4$. The following reasoning holds for all $n \geq 4$ since $n! - 1 > 1$ for all $n \geq 4$. Consider the fact that $(n+1)! - 2, (n+1)! - 3, \ldots, (n+1)! - n, (n+1)! - (n+1)$ will always be composite, since values $2, 3, \ldots, n, n+1$ can be factored from each term respectively. Therefore, there exists n consecutive composite numbers. Since there exists four consecutive composite Fibonacci numbers, we can find one, two and three consecutive composite Fibonacci numbers within the four consecutive composite Fibonacci numbers. Hence we have proved that there exists n consecutive composite Fibonacci numbers for all $n \geq 1$. $\square$

Example 7.28. We provide an example for the statement of Proposition 7.27 below. Let $n = 5$. Then, there should exist at least 5 consecutive composite Fibonacci numbers. In particular, it follows from our reasoning that $F_{6!-2}, F_{6!-3}, F_{6!-4}, F_{6!-5}$ and $F_{6!-6}$ are all composite. We prove that these numbers are composite by listing their prime factorizations below:

$$
\begin{aligned}
F_{6!-2} = F_{718} = {} & 719 \cdot 1648529 \cdot 1517456267839 \cdot \\
& 5910458660850425063891050543618811245979406031793917 91 \cdot \\
& 4754204377346982207473680271667493829277014170165571936622687163769354762 41 \\
F_{6!-3} = F_{717} = {} & 2 \cdot 1433 \cdot 10037 \cdot 62141 \cdot 7099733 \cdot 2228536579597318057 \cdot \\
& 15634731455464012955341 \cdot 2854690886229614923336 9 \cdot \\
& 247447693935443076085383201164597693878632846428321319838397619330 49 \\
F_{6!-4} = F_{716} = {} & 3 \cdot 359 \cdot 21481 \cdot 156089 \cdot 316590102769 \cdot \\
& 1066737847220321 \cdot 281867565966225823 69 \cdot 669322542794846474 41 \cdot \\
& 3418816640903898929534613769 \cdot 245420883798341583087107571814316523674222 41 \\
F_{6!-5} = F_{715} = {} & 5 \cdot 89 \cdot 233 \cdot 661 \cdot 8581 \cdot 474541 \cdot 7096612381 \cdot \\
& 14736206161 \cdot 1929584153756850496621 \cdot 196418919424255540016736161 \cdot \\
& 1078611204181410605710773927691692704779513293013309045560198946 1 \\
F_{6!-6} = F_{714} = {} & 2^3 \cdot 13 \cdot 29 \cdot 211 \cdot 239 \cdot 421 \cdot 919 \cdot 1429 \cdot 1597 \cdot 3469 \cdot 3571 \cdot 10711 \cdot \\
& 258469 \cdot 6376021 \cdot 27932732439809 \cdot 1595129398158557881 21 \cdot \\
& 27653866239836258463881623092961 \cdot 2037962186691204100928530687800099843828 1
\end{aligned}
$$

Proposition 7.29 (Exercise 18 from 14.2 [1]). *Prove that $9 \mid F_{n+24}$ if and only if $9 \mid F_n$.*

Proof. Assume that $9 \mid F_{n+24}$.

WWTS: $9 \mid F_n$.

By Proposition 4.10, we know $F_{n+24} = F_{n-1}F_{24} + F_nF_{25}$. Hence we have the following implication:

$$9 \mid F_{n+24} \implies 9 \mid F_{n-1}F_{24} + F_nF_{25}.$$

Observe that $F_{24} = 46368 = 2^5 \cdot 3^2 \cdot 7 \cdot 23$, and hence $9 \mid F_{24}$. So clearly $9 \mid F_{n+1}F_{24}$ also. Thus $F_{n-1}F_{24} = 9K$ for some $K \in \mathbb{Z}$. So we can write

$$9 \mid F_{n-1}F_{24} + F_nF_{25} \implies 9 \mid 9K + F_nF_{25}.$$

So by Lemma 6.5, we have $9 \mid F_nF_{25}$. But since $F_{25} = 5^2 \cdot 3001$, then clearly 9 does not divide F_{25}. So we have $9 \mid F_nF_{25}$ and $\gcd(9, F_{25}) = 1$, and hence by Lemma 6.6 the result $F \mid F_9$ follows.

Now assume that $9 \mid F_9$.

WWTS: $9 \mid F_{n+24}$.

Since $9 \mid F_n$ then $F_n = 9K$ for some $K \in \mathbb{Z}$. Also since $F_{24} = 46368 = 2^5 \cdot 3^2 \cdot 7 \cdot 23$, then $F_{24} = 9J$ for some $J \in \mathbb{Z}$. Observe the following sequence of equalities:

$$\begin{aligned} F_{n+24} &= F_{n-1}F_{24} + F_nF_{25} && \text{by Proposition 4.10} \\ &= F_{n-1}F_{24} + 9K \cdot F_{25} && \text{since } 9 \mid F_n \\ &= F_{n-1} \cdot 9J + 9K \cdot F_{25} && \text{since } 9 \mid F_{24} \\ &= 9(F_{n-1} \cdot J + F_{25} \cdot K). \end{aligned}$$

Since $F_{n-1} \cdot J + F_{25} \cdot K$ is an integer, we conclude that $9 \mid F_{n+24}$ as desired. □

Proposition 7.30 (Exercise 19 from 14.2 [1]). *Use induction to show that for all $n \geq 1$, we have $F_{2n} \equiv n(-1)^{n+1} \pmod 5$.*

Proof. We induct on n

Base Cases: $(n = 1)$ **and** $(n = 2)$
$(n = 1)$ Observe that $F_{2(1)} = F_2 = 1$ and $1 \cdot (-1)^{1+1} = 1 \cdot (-1)^2 = 1$. Hence $F_{2(1)} \equiv 1(-1)^{1+1} \pmod 5$. ✓
$(n = 2)$ Observe that $F_{2(2)} = F_4 = 3$ and $2(-1)^{2+1} = 2(-1)^3 = -2$. Since $3 \equiv -2 \pmod 5$, then $F_{2(2)} \equiv 2(-1)^{2+1} \pmod 5$. ✓

Induction Hypotheses: $F_{2(k)} \equiv k(-1)^{k+1} \pmod 5$ and $F_{2(k-1)} \equiv (k-1)(-1)^{(k-1)+1}$ for some $k \geq 2$.

WWTS: $F_{2(k+1)} \equiv (k+1)(-1)^{k+2} \pmod 5$.

$$\begin{aligned}
F_{2k+2} &= F_{2k} + F_{2k+1} && \text{by Equation (1)}\\
&= F_{2k} + (F_{2k} + F_{2k-1}) && \text{by Equation (1)}\\
&= 2F_{2k} + F_{2k-1}\\
&= 2F_{2k} + (F_{2k} - F_{2k-2}) && \text{by Equation (3)}\\
&= 3F_{2k} - F_{2k-2}\\
&\equiv 3(k(-1)^{k+1}) - (k-1)(-1)^{k} \pmod 5 && \text{by Induction Hypotheses}\\
&\equiv (-1)(-1)\cdot 3k(-1)^{k+1} - (-1)(-1)(k-1)(-1)^{(k-1)+1} \pmod 5\\
&\equiv (-1)\cdot 3k(-1)^{(k+1)+1} - (k-1)(-1)^{(k+1)+1} \pmod 5\\
&\equiv (-3k-(k-1))(-1)^{(k+1)+1} \pmod 5\\
&\equiv (-4k+1)(-1)^{(k+1)+1} \pmod 5\\
&\equiv (k+1)(-1)^{(k+1)+1} \pmod 5.
\end{aligned}$$

Hence $F_{2n} \equiv n(-1)^{n+1} \pmod 5$ for all $n \geq 1$. □

Proposition 7.31 (Exercise 20 from 14.2 [1])**.** *Derive the identity $F_{n+3} = 3F_{n+1} - F_{n-1}$ for all $n \geq 1$.*

Proof. Let $n \geq 1$ be given.

WWTS: $F_{n+3} = 3F_{n+1} - F_{n-1}$.

Notice the following:

$$\begin{aligned}
F_{n+3} &= F_{n-1}F_3 + F_nF_4 && \text{by Proposition 4.10}\\
&= 2F_{n-1} + 3F_n && \text{since } F_3 = 2 \text{ and } F_4 = 3\\
&= F_{n-1} + F_{n-1} + F_n + F_n + F_n\\
&= 2F_{n+1} + F_n && \text{by Equation (1)}\\
&= 2F_{n+1} + (F_{n+1} - F_{n-1}) && \text{by Equation (3)}\\
&= 3F_{n+1} - F_{n-1}.
\end{aligned}$$

Hence it proven that $F_{n+3} = 3F_{n+1} - F_{n-1}$ for all $n \geq 2$. □

Exercises from Section 14.3 from Burton

Proposition 7.32 (Exercise 3a from 14.3 [1])**.** *For all $n \geq 2$, the following identity holds:*

$$F_{2n-1} = F_n^2 + F_{n-1}^2.$$

Proof. Let $n \geq 2$ be given.

WWTS: $F_{2n-1} = F_n^2 + F_{n-1}^2$.

$$\begin{aligned} F_{2n-1} &= F_{n+(n-1)} \\ &= F_{n-1}F_{n-1} + F_nF_n && \text{by Proposition 4.10} \\ &= F_{n-1}^2 + F_n^2. \end{aligned}$$

Hence, we may conclude $F_{2n-1} = F_n^2 + F_{n-1}^2$ for all $n \geq 2$. □

Proposition 7.33 (Exercise 3b from 14.3 [1])**.** *For all $n \geq 2$, the following identity holds:*

$$F_{2n} = F_{n+1}^2 - F_{n-1}^2.$$

Proof. Let $n \geq 2$ be given.

WWTS: $F_{2n} = F_{n+1}^2 - F_{n-1}^2$.

$$\begin{aligned} F_{2n} &= F_{n+n} \\ &= F_{n-1}F_n + F_nF_{n+1} && \text{by Proposition 4.10} \\ &= (F_{n-1} + F_{n+1}) \cdot F_n \\ &= (F_{n-1} + F_{n+1})(F_{n+1} - F_{n-1}) && \text{by Equation (3)} \\ &= F_{n+1}^2 - F_{n-1}^2. \end{aligned}$$

Hence, we may conclude $F_{2n} = F_{n+1}^2 - F_{n-1}^2$ for all $n \geq 2$. □

Remark 7.34. Notice that the two propositions above coincide with Corollaries 4.12 and 4.14 from an earlier section.

Proposition 7.35 (Exercise 4a from 14.3 [1])**.** *For all $n \geq 3$, the following identity holds:*

$$F_{n+1}^2 + F_{n-2}^2 = 2F_{2n-1}.$$

Proof. Let $n \geq 3$ be given.

WWTS: $F_{n+1}^2 + F_{n-2}^2 = 2F_{2n-1}$.

Observe the following sequence of equalities:

$$\begin{aligned}
F_{n+1}^2 + F_{n-2}^2 &= (F_n + F_{n-1})^2 + F_{n-2}^2 && \text{by Equation (1)}\\
&= F_n^2 + 2F_nF_{n-1} + F_{n-2}^2 + F_{n-1}^2\\
&= (F_{2n-1} - F_{n-1}^2) + 2F_nF_{n-1} + F_{n-2}^2 + F_{n-1}^2 && \text{by Proposition 7.32}\\
&= F_{2n-1} + 2F_nF_{n-1} + F_{n-2}^2\\
&= F_{2n-1} + 2F_nF_{n-1} + (F_n - F_{n-1})^2 && \text{by Equation (4)}\\
&= F_{2n-1} + 2F_nF_{n-1} + (F_n^2 - 2F_nF_{n-1} + F_{n-1}^2)\\
&= F_{2n-1} + F_n^2 + F_{n-1}^2\\
&= F_{2n-1} + F_{2n-1} && \text{by Proposition 7.32}\\
&= 2F_{2n-1}.
\end{aligned}$$

Hence, we may conclude $F_{n+1}^2 + F_{n-2}^2 = 2F_{2n-1}$ for all $n \geq 3$. □

Proposition 7.36 (Exercise 4b from 14.3 [1])**.** *For all $n \geq 3$, the following identity holds:*

$$F_{n+2}^2 + F_{n-1}^2 = 2(F_n + F_{n+1}^2).$$

Proof. Let $n \geq 3$ be given.

WWTS: $F_{n+2}^2 + F_{n-1}^2 = 2(F_n + F_{n+1}^2)$.

Consider the following sequence of equalities:

$$\begin{aligned}
F_{n+2}^2 + F_{n-1}^2 &= 2F_{2(n+1)-1} && \text{by Proposition 7.35}\\
&= 2(F_{n+1}^2 + F_n^2) && \text{by Proposition 7.32.}
\end{aligned}$$

Hence $F_{n+2}^2 + F_{n-1}^2 = 2(F_n^2 + F_{n+1}^2)$ for all $n \geq 2$. □

Proposition 7.37 (Exercise 6a from 14.3 [1])**.** *For all $n \geq 2$, the following identity holds:*

$$F_{n+1}^2 - 4F_nF_{n-1} = F_{n-2}^2.$$

Proof. Let $n \geq 2$ be given.

WWTS: $F_{n+1}^2 - 4F_nF_{n-1} = F_{n-2}^2$.

Observe the following sequence of equalities:

$$\begin{aligned}
F_{n+1}^2 - 4F_nF_{n-1} &= (F_n + F_{n-1})^2 - 4F_nF_{n-1} && \text{by Equation (1)} \\
&= (F_n^2 + 2F_nF_{n-1} + F_{n-1}^2) - 4F_nF_{n-1} \\
&= F_n^2 - 2F_nF_{n-1} + F_{n-1}^2 \\
&= (F_n - F_{n-1})^2 \\
&= F_{n-2}^2.
\end{aligned}$$

Hence $F_{n+1}^2 - 4F_nF_{n-1} = F_{n-2}^2$ holds for all $n \geq 2$. □

Proposition 7.38 (Exercise 6b from 14.3 [1])**.** *For all $n \geq 3$, the following identity holds:*

$$F_{n+1}F_{n-1} - F_{n+2}F_{n-2} = 2(-1)^n.$$

Proof. Let $n \geq 3$ be given.

WWTS: $F_{n+1}F_{n-1} - F_{n+2}F_{n-2} = 2(-1)^n.$

Consider the following sequence of equalities:

$$\begin{aligned}
&F_{n+1}F_{n-1} - F_{n+2}F_{n-2} \\
&\qquad = F_{n+1}F_{n-1} - (F_{n+1} + F_n)(F_n - F_{n-1}) && \text{by Equations (1) and (4)} \\
&\qquad = F_{n+1}F_{n-1} - (F_{n+1}F_n - F_{n+1}F_{n-1} + F_n^2 - F_nF_{n-1}) \\
&\qquad = F_{n+1}F_{n-1} - \big(F_n(F_{n+1} - F_{n-1}) - F_{n+1}F_{n-1} + F_n^2\big) \\
&\qquad = F_{n+1}F_{n-1} - (F_n^2 - F_{n+1}F_{n-1} + F_n^2) && \text{by Equation (3)} \\
&\qquad = 2F_{n+1}F_{n-1} - 2F_n^2 \\
&\qquad = 2(F_{n+1}F_{n-1} - F_n^2) \\
&\qquad = 2\Big(F_{n+1}F_{n-1} - \big(F_{n+1}F_{n-1} + (-1)^{n-1}\big)\Big) && \text{by Proposition 4.16} \\
&\qquad = 2\big(-(-1)^{n-1}\big) \\
&\qquad = 2(-1)^n.
\end{aligned}$$

Hence, we may conclude $F_{n+1}F_{n-1} - F_{n+2}F_{n-2} = 2(-1)^n$ for all $n \geq 3$. □

Proposition 7.39 (Exercise (6c) from 14.3 [1])**.** *For all $n \geq 3$, the following identity holds:*

$$F_n^2 - F_{n+2}F_{n-2} = (-1)^n.$$

Proof. Let $n \geq 3$ be given.

WWTS: $F_n^2 - F_{n+2}F_{n-2} = (-1)^n$.

By Proposition 4.16, the following holds:

$$F_n^2 = F_{n+1}F_{n-1} + (-1)^{n-1}.$$

Multiplying this identity by -1 it follows that

$$-F_n^2 = -F_{n+1}F_{n-1} + (-1)^n$$

Hence

$$-F_n^2 + F_{n+1}F_{n-1} = (-1)^n.$$

Thus it suffices to show

$$-F_n^2 + F_{n+1}F_{n-1} = F_n^2 - F_{n+2}F_{n-2}.$$

Consider the following sequence of equalities:

$$\begin{aligned}
-F_n^2 + F_{n+1}F_{n-1} &= -F_n^2 + F_{n-1}(F_{n+2} - F_n) && \text{by Equation (3)}\\
&= -F_n^2 + F_{n-1}F_{n+2} - F_{n-1}F_n\\
&= -F_n(F_n + F_{n-1}) + F_{n-1}F_{n+2}\\
&= -F_nF_{n+1} + F_{n-1}F_{n+2} && \text{by Equation (1)}\\
&= -F_nF_{n+1} + (F_n - F_{n-2})F_{n+2} && \text{by Equation (3)}\\
&= -F_nF_{n+1} + F_nF_{n+2} - F_{n-2}F_{n+2}\\
&= F_n(-F_{n+1} + F_{n+2}) - F_{n-2}F_{n+2}\\
&= F_n(F_n) - F_{n-2}F_{n+2} && \text{by Equation (4)}\\
&= F_n^2 - F_{n-2}F_{n+2}.
\end{aligned}$$

Therefore $F_n^2 - F_{n+2}F_{n-2} = (-1)^n$ for all $n \geq 3$. □

Proposition 7.40 (Exercise 6e from 14.3 [1])**.** *For all $n \geq 1$, the following identity holds:*

$$F_nF_{n+1}F_{n+3}F_{n+4} = F_{n+2}^4 - 1.$$

Proof. Let $n \geq 1$ be given.

WWTS: $F_nF_{n+1}F_{n+3}F_{n+4} = F_{n+2}^4 - 1$.

Observe the following sequence of equalities:

$$
\begin{aligned}
&F_nF_{n+1}F_{n+3}F_{n+4} \\
&\qquad = (F_{n+4}F_n)\cdot(F_{n+1}F_{n+3}) \\
&\qquad = (F_{n+2}^2-(-1)^{n+2})\cdot(F_{n+1}F_{n+3}) && \text{by Proposition 7.39} \\
&\qquad = (F_{n+2}^2-(-1)^{n+2})\cdot(F_{n+2}^2-(-1)^{n+1}) && \text{by Proposition 4.16} \\
&\qquad = (F_{n+2}^2+(-1)^{n+3})\cdot(F_{n+2}^2+(-1)^{n+2}) \\
&\qquad = F_{n+2}^4+(-1)^{n+3}F_{n+2}^2+(-1)^{n+2}F_{n+2}^2+(-1)^{2n+5} \\
&\qquad = F_{n+2}^4+F_{n+2}^2\cdot((-1)^{n+3}+(-1)^{n+2})-1 && \text{since } (-1)^{2n+5}=-1 \\
&\qquad = F_{n+2}^4+F_{n+2}^2\cdot 0-1 && \text{since } (-1)^{n+3}+(-1)^{n+2}=0 \\
&\qquad = F_{n+2}^4-1.
\end{aligned}
$$

Hence $F_nF_{n+1}F_{n+3}F_{n+4}=F_{n+2}^4-1$ holds for all $n\geq 1$. □

Proposition 7.41 (Exercise 11 from 14.3 [1])**.** *For all $n\geq 1$, the following identity holds:*

$$F_{2n+2}F_{2n-1}-F_{2n}F_{2n+1}=1.$$

Proof. Let $n\geq 1$ be given.

WWTS: $F_{2n+2}F_{2n-1}-F_{2n}F_{2n+1}=1.$

Consider the following sequence of equalities:

$$
\begin{aligned}
&F_{2n+2}F_{2n-1}-F_{2n}F_{2n+1} = \frac{\alpha^{2n+2}-\beta^{2n+2}}{\alpha-\beta}\cdot\frac{\alpha^{2n-1}-\beta^{2n-1}}{\alpha-\beta}-\frac{\alpha^{2n}-\beta^{2n}}{\alpha-\beta}\cdot\frac{\alpha^{2n+1}-\beta^{2n+1}}{\alpha-\beta} \\
&= \frac{(\alpha^{2n+2}-\beta^{2n+2})\cdot(\alpha^{2n-1}-\beta^{2n-1})}{(\alpha-\beta)^2}-\frac{(\alpha^{2n}-\beta^{2n})\cdot(\alpha^{2n+1}-\beta^{2n+1})}{(\alpha-\beta)^2} \\
&= \frac{(\alpha^{2n+2}-\beta^{2n+2})\cdot(\alpha^{2n-1}-\beta^{2n-1})-(\alpha^{2n}-\beta^{2n})\cdot(\alpha^{2n+1}-\beta^{2n+1})}{(\alpha-\beta)^2} \\
&= \frac{(\alpha^{4n+1}+\beta^{4n+1}-\alpha^{2n+2}\beta^{2n-1}-\alpha^{2n-1}\beta^{2n+2})-(\alpha^{4n+1}+\beta^{4n+1}-\alpha^{2n}\beta^{2n+1}-\alpha^{2n+1}\beta^{2n})}{(\alpha-\beta)^2} \\
&= \frac{\alpha^{4n+1}+\beta^{4n+1}-\alpha^{2n+2}\beta^{2n-1}-\alpha^{2n-1}\beta^{2n+2}-\alpha^{4n+1}-\beta^{4n+1}+\alpha^{2n}\beta^{2n+1}+\alpha^{2n+1}\beta^{2n}}{(\alpha-\beta)^2} \\
&= \frac{\cancel{\alpha^{4n+1}+\beta^{4n+1}}-\alpha^{2n+2}\beta^{2n-1}-\alpha^{2n-1}\beta^{2n+2}\cancel{-\alpha^{4n+1}-\beta^{4n+1}}+\alpha^{2n}\beta^{2n+1}+\alpha^{2n+1}\beta^{2n}}{(\alpha-\beta)^2} \\
&= \frac{-\alpha^{2n+2}\beta^{2n-1}-\alpha^{2n-1}\beta^{2n+2}+\alpha^{2n}\beta^{2n+1}+\alpha^{2n+1}\beta^{2n}}{(\alpha-\beta)^2}
\end{aligned}
$$

$$= \frac{-\alpha^3(\alpha^{2n-1}\beta^{2n-1}) - \beta^3(\alpha^{2n-1}\beta^{2n-1}) + \beta(\alpha^{2n}\beta^{2n}) + \alpha(\alpha^{2n}\beta^{2n})}{(\alpha-\beta)^2}$$
$$= \frac{-\alpha^3(\alpha\beta)^{2n-1} - \beta^3(\alpha\beta)^{2n-1} + \beta(\alpha\beta)^{2n} + \alpha(\alpha\beta)^{2n}}{(\alpha-\beta)^2}$$
$$= \frac{\alpha^3 + \beta^3 + \beta + \alpha}{(\alpha-\beta)^2}$$
$$= \frac{4+1}{(\sqrt{5})^2}$$
$$= \frac{5}{5}$$
$$= 1.$$

Hence $F_{2n+2}F_{2n-1} - F_{2n}F_{2n+1} = 1$ for all $n \geq 1$. □

Proposition 7.42 (Exercise 15 from 14.3 [1]). *Prove that the sum of any 20 consecutive Fibonacci numbers is divisible by F_{10}.*

Proof. Consider the following for all $n \geq 20$:

$$F_1 + F_2 + F_3 \cdots + F_{19} + \cdots + F_{n-1} + F_n.$$

Then the following holds:

$$\sum_{i=1}^{n} F_i - \sum_{i=1}^{n-20} F_i = \sum_{i=n-19}^{n} F_i.$$

From Proposition 4.1, it follows that

$$(F_{n+2} - 1) - (F_{n-18} - 1) = F_{n+2} - F_{n-18} = \sum_{i=n-19}^{n} F_i.$$

Therefore, any sum of twenty consecutive Fibonacci numbers starting at F_{n-19} and ending at F_n is equal to $F_{n+2} - F_{n-18}$ for all $n \geq 20$. Hence, in order to prove that the sum of any consecutive Fibonacci numbers is divisible by F_{10}, it suffices to show that $F_{n+2} - F_{n-18}$ is divisible by F_{10} for all $n \geq 20$. We prove this result, via induction, below.

We induct on n, in order to prove that $F_{n+2} - F_{n-18} \equiv 0 \pmod{F_{10}}$ for all $n \geq 20$.

Base Cases: $(n = 20)$ and $(n = 21)$
$(n = 20)$ Observe that $F_{22} - F_2 = 17711 - 1 = 17710 = 55 \cdot 322 \equiv 0 \pmod{55}$.
$(n = 21)$ Observe that $F_{23} - F_3 = 28657 - 2 = 28655 = 55 \cdot 521 \equiv 0 \pmod{55}$.
Since $F_{10} = 55$, both of our Base Cases are proven. ✓

Induction Hypotheses: Suppose the following for some $k \geq 21$:
$F_{k+2} - F_{k-18} \equiv 0 \pmod{F_{10}}$ and $F_{(k-1)+2} - F_{(k-1)-18} \equiv 0 \pmod{F_{10}}$.

WWTS: $F_{(k+1)+2} - F_{(k+1)-18} \equiv 0 \pmod{F_{10}}$.

Consider the following sequence of equalities:

$$\begin{aligned}
F_{(k+1)+2} - F_{(k+1)-18} &= F_{k+3} - F_{k-17} \\
&= F_{k+2} + F_{k+1} - F_{k-18} - F_{k-19} && \text{by Equation (1)} \\
&= (F_{k+2} - F_{k-18}) + (F_{k+1} - F_{k-19}) \\
&\equiv 0 + 0 \pmod{F_{10}} && \text{by Induction Hypotheses} \\
&\equiv 0 \pmod{F_{10}}.
\end{aligned}$$

Hence the sum of any twenty consecutive Fibonacci numbers is divisible by F_{10}. □

Proposition 7.43 (Exercise 16 from 14.3 [1])**.** *For all $n \geq 4$ prove that $F_n + 1$ is not a prime. In particular, the following four identities hold:*

$$\begin{aligned}
F_{4k} + 1 &= F_{2k-1}(F_{2k} + F_{2k+2}) \\
F_{4k+1} + 1 &= F_{2k+1}(F_{2k-1} + F_{2k+1}) \\
F_{4k+2} + 1 &= F_{2k+2}(F_{2k+1} + F_{2k-1}) \\
F_{4k+3} + 1 &= F_{2k+1}(F_{2k+1} + F_{2k+3}).
\end{aligned}$$

Proof. In order to prove that $F_n + 1$ is never prime for all $n \geq 4$, we establish the following identities:

$$F_{4k} + 1 = F_{2k-1}(F_{2k} + F_{2k+2})$$

$$F_{4k+1} + 1 = F_{2k+1}(F_{2k-1} + F_{2k+1})$$

$$F_{4k+2} + 1 = F_{2k+2}(F_{2k+1} + F_{2k-1})$$

$$F_{4k+3} + 1 = F_{2k+1}(F_{2k+1} + F_{2k+3}).$$

We begin by proving the first identity within the following sequence of equalities:

$$\begin{aligned}
F_{4k} + 1 &= F_{2k+2k} + 1 \\
&= F_{2k-1}F_{2k} + F_{2k}F_{2k+1} + 1 && \text{by Proposition 4.10} \\
&= F_{2k-1}F_{2k} + F_{2k}F_{2k+1} + 1 \\
&= F_{2k-1}F_{2k} + F_{2k-1}F_{2k+2} && \text{by Proposition 7.41} \\
&= F_{2k-1}(F_{2k} + F_{2k+2})
\end{aligned}$$

Hence we have established that $F_{4k} + 1 = F_{2k-1}(F_{2k} + F_{2k+2})$ for all $n \geq 4$. Next we establish that $F_{4k+1} + 1 = F_{2k+1}(F_{2k-1} + F_{2k+1})$ for all $n \geq 4$.

$$F_{4k+1} + 1 = F_{2(2k+1)-1} + 1$$

$$
\begin{aligned}
&= F_{2k}^2 + F_{2k+1}^2 + 1 && \text{by Proposition 7.32}\\
&= F_{2k}^2 + 1 + F_{2k+1}^2\\
&= F_{2k+1}F_{2k-1} + F_{2k+1}^2 && \text{by Proposition 4.16}\\
&= F_{2k+1}(F_{2k-1} + F_{2k+1}).
\end{aligned}
$$

Hence we have established that $F_{4k+1} + 1 = F_{2k+1}(F_{2k-1} + F_{2k+1}$ for all $n \geq 4$. Next we establish that $F_{4k+2} + 1 = F_{2k+2}(F_{2k+1} + F_{2k-1})$ for all $n \geq 4$.

$$
\begin{aligned}
F_{4k+2} + 1 &= F_{(2k+1)+(2k+1)} + 1\\
&= F_{2k} + F_{2k+1} + F_{2k+1}F_{2k+2} + 1 && \text{by Propostion 4.10}\\
&= F_{2k+2}F_{2k+1} + F_{2k+2}F_{2k-1} && \text{by Proposition 7.41.}\\
&= F_{2k+2}(F_{2k+1} + F_{2k-1}).
\end{aligned}
$$

Hence we have established that $F_{4k+2} + 1 = F_{2k+2}(F_{2k+1} + F_{2k-1})$ for all $n \geq 4$. Next we establish that $F_{4k+3} + 1 = F_{2k+1}(F_{2k+1} + F_{2k+3})$ for all $n \geq 4$.

$$
\begin{aligned}
F_{4k+3} + 1 &= F_{4k+4-1} + 1\\
&= F_{2(2k+2)-1} + 1\\
&= F_{2k+1}^2 + F_{2k+2}^2 + 1 && \text{by Proposition 7.32}\\
&= F_{2k+1}^2 + F_{2k+1}F_{2k+3} && \text{by Proposition 4.16}\\
&= F_{2k+1}(F_{2k+1} + F_{2k+3}).
\end{aligned}
$$

Hence we have established that $F_{4k+3} + 1 = F_{2k+1}(F_{2k+1} + F_{2k+3})$ for all $n \geq 4$. Therefore, we have proven that for all $n \geq 4$, the number $F_n + 1$ is never prime, since all four identities are established. □

Remark 7.44. The diagonal sums of Pascal's triangle yield the Fibonacci numbers in the following sense as the diagram below implies.

	1												
$n=1$	1	1											
$n=2$	1	2	1										
$n=3$	1	3	3	1									
$n=4$	1	4	6	4	1								
$n=5$	1	5	10	10	5	1							
$n=6$	1	6	15	20	15	6	1						
$n=7$	1	7	21	35	35	21	7	1					
$n=8$	1	8	28	56	70	56	28	8	1				
$n=9$	1	9	36	84	126	126	84	36	9	1			
$n=10$	1	10	45	120	210	252	210	120	45	10	1		
$n=11$	1	11	55	165	330	462	462	330	165	55	11	1	
$n=12$	1	12	66	220	495	792	924	792	495	220	66	12	1

In Proposition 7.45 below, we give a proof of this observation.

Proposition 7.45 (Exercise 23 from 14.3 [1])**.** *The following gives a formula for the Fibonacci numbers in terms of the binomial coefficients:*

$$F_n = \binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \cdots + + \binom{n-j}{j-1} + \binom{n-j-1}{j}.$$

Proof. $\square$

8 Lucas Identities from Burton

Proposition 8.1 (Exercise 17a from 14.3 [1]). *For all $n \geq 1$, the following identity holds:*

$$L_1 + L_2 + L_3 + \cdots L_n = L_{n+2} - 3.$$

Proof. We induct on n.

Base Case: $(n = 1)$ Observe that $L_{1+2} - 3 = L_3 - 3 = 4 - 3 = 1 = L_1$.

Induction Hypothesis: Assume $L_1 + L_2 + \cdots + L_k + L_{k+2} = L_{k+3} - 3$ for some $k \geq 1$.

WWTS: $L_1 + L_2 + \cdots + L_k + L_{k+1} = L_{k+3} - 3.$

$$\begin{aligned} L_1 + L_2 + \cdots + L_k + L_{k+1} &= (L_1 + L_2 + \cdots + L_k) + L_{k+1} \\ &= L_{k+2} - 3 + L_{k+2} && \text{by Induction Hypothesis} \\ &= L_{k+2} + L_{k+1} - 3 \\ &= L_{k+3} - 3 && \text{by Equation (5).} \end{aligned}$$

Hence, we may conclude $L_1 + L_2 + \cdots + L_k + L_{k+1} = L_{k+3} - 3$ for all $n \geq 1$. □

Proposition 8.2 (Exercise 17b from 14.3 [1]). *For all $n \geq 1$, the following identity holds:*

$$L_1 + L_3 + L_5 + \cdots + L_{2n-1} = L_{2n} - 2.$$

Proof. Let $n \geq 1$ be given.

WWTS: $L_1 + L_3 + L_5 + \cdots + L_{2n-1} = L_{2n} - 2.$

Observe the following sequence of equalities:

$$\begin{aligned} & L_1 + L_3 + L_5 + \cdots + L_{2n-1} \\ &\quad = (F_1 + 2F_0) + (F_3 + 2F_2) + \cdots + (F_{2n-1} + 2F_{2n+2}) && \text{by Proposition 8.8} \\ &\quad = (F_0 + F_1 + F_2 + \cdots + F_{2n-1}) + (F_0 + F_2 + F_4 + \cdots + F_{2n-2}) \\ &\quad = (F_{(2n-1)+2} - 1) + (F_{(2n-1)+1} - 1) && \text{by Proposition 4.1 and 4.3} \\ &\quad = F_{2n+1} + F_{2n-1} - 2 \\ &\quad = L_{2n} - 2 && \text{by Proposition 8.8.} \end{aligned}$$

Hence $L_1 + L_3 + L_5 + \cdots + L_{2n-1} = L_{2n} - 2$ holds for all $n \geq 1$. □

Proposition 8.3 (Exercise 17c from 14.3 [1]). *For all $n \geq 1$, the following identity holds::*

$$L_2 + L_4 + L_6 + \cdots + L_{2n} = L_{2n+1} - 1.$$

Proof. Let $n \geq 1$ be given. We induct on n.

Base Case: ($n = 1$) Notice the left hand side of our statement as the following: $L_2 = 3$. Notice the right hand side of our statement as the following: $L_{2+1} - 1$. Observe that $L_3 - 1 = 4 - 1 = 3$. Thus our base case is proven. ✓

Induction Hypothesis: Assume $L_2 + L_4 + \cdots + L_{2k} = L_{2k+1} - 1$ for some $k \geq 1$.

WWTS: $L_2 + L_4 + \cdots + L_{2n} + L_{2n+2} = L_{2(n+1)+1} - 1.$

Observe the following sequence of equalities:

$$\begin{aligned} L_2 + L_4 + \cdots + L_2k + L_{2k+2} &= (L_2 + L_4 + \cdots + L_{2k}) + L_{2k+2} \\ &= L_{2k+1} - 1 + L_{2k+2} && \text{by Induction Hypothesis} \\ &= L_{2k+1} + L_{2k+2} - 1 \\ &= L_{2k+3} - 1 && \text{by Definition 5.3.} \end{aligned}$$

Hence, we may conclude that $L_2 + L_4 + L_6 + \cdots + L_{2n} = L_{2n+1} - 1$. for all $n \geq 1$. □

Proposition 8.4 (Exercise 17d from 14.3 [1]). *For all $n \geq 2$, the following identitiy holds:*

$$L_n^2 = L_{n+1}L_{n-1} + 5(-1)^n.$$

Proof. We induct on n.

Base Case: ($n = 2$) Observe that $L_{2+1}L_{2-1} + 5(-1)^2 = L_3L_1 + 5 = 4 \cdot 1 + 5 = 9 = L_2^2$. ✓

Induction Hypothesis: Assume $L_k^2 = L_{k+1}L_{k-1} + 5(-1)^k$ for some $k \geq 2$.

WWTS: $L_{k+1}^2 = L_{k+2}L_k + 5(-1)^{k+1}$.

Observe the following sequence of equalities.

$$\begin{aligned} L_{k+2}L_k + 5(-1)^{k+1} &= (L_{k+1} + L_k)L_k + 5(-1)^{k+1} && \text{by Equation (5)} \\ &= L_kL_{k+1} + L_k^2 + 5(-1)^{k+1} \\ &= L_kL_{k+1} + L_{k+1}L_{k-1} + 5(-1)^k + 5(-1)^{k+1} && \text{by Induction Hypothesis} \end{aligned}$$

$$
\begin{aligned}
&= L_{k+1}(L_k + L_{k-1}) \\
&= L_{k+1}L_{k+1} && \text{by Equation (5)} \\
&= L_{k+1}^2.
\end{aligned}
$$

Hence, we may conclude $L_{k+1}^2 = L_{k+2}L_k + 5(-1)^{k+1}$ for all $n \geq 2$. □

Proposition 8.5 (Exercise 17e from 14.3 [1])**.** *For all $n \geq 1$, the following identity holds:*

$$L_1^2 + L_2^2 + L_3^2 + \cdots + L_n^2 = L_nL_{n+1} - 2.$$

Proof. Let $n \geq 1$ be given.

WWTS: $L_1^2 + L_2^2 + L_3^2 + \cdots + L_n^2 = L_nL_{n+1} - 2.$

Observe the following sequence of equalities:

$$
\begin{aligned}
&L_1^2 + L_2^2 + L_3^2 + \cdots + L_n^2 \\
&\qquad = \sum_{k=1}^{n} \left(\alpha^k + \beta^k\right)^2 \\
&\qquad = \sum_{k=1}^{n} \alpha^{2k} + \beta^{2k} + 2(\alpha\beta)^k \\
&\qquad = \sum_{k=1}^{n} L_{2k} + 2(-1)^k && \text{since } \alpha\beta = -1 \\
&\qquad = \sum_{k=1}^{n} L_{2k} + 2\sum_{k=1}^{n}(-1)^k \\
&\qquad = (L_{2n+1} - 1) + 2\sum_{k=1}^{n}(-1)^k && \text{by Proposition 8.3} \\
&\qquad = (L_{2n+1} - 1) + \begin{cases} 0 & \text{if } n \text{ is even} \\ -2 & \text{if } n \text{ is odd} \end{cases} \\
&\qquad = (L_nL_{n+1} - (-1)^n) - 1 + \begin{cases} 0 & \text{if } n \text{ is even} \\ -2 & \text{if } n \text{ is odd} \end{cases} && \text{by Proposition 8.16} \\
&\qquad = L_nL_{n+1} - 2,
\end{aligned}
$$

where the last equality holds since $-(-1)^n - 1 + 0 = 2$ regardless of n being even or odd. Hence $L_1^2 + L_2^2 + L_3^2 + \cdots + L_n^2 = L_nL_{n+1} - 2$ holds for all $n \geq 1$. □

Proposition 8.6 (Exercise 17f from 14.3 [1]). *For all $n \geq 2$, the following identity holds:*

$$L_{n+1}^2 - L_n^2 = L_{n-1}L_{n+2}.$$

Proof. Let $n \geq 2$ be given.

WWTS: $L_{n+1}^2 - L_n^2 = L_{n-1}L_{n+2}$.

Consider the following sequence of equalities:

$$\begin{aligned}
L_{n+1}^2 - L_n^2 &= (\alpha^{n+1} + \beta^{n+1})^2 - (\alpha^n + \beta^n)^2 \\
&= \alpha^{2n+2} + 2\alpha^{n+1}\beta^{n+1} + \beta^{2n+2} - (\alpha^{2n} + 2\alpha^n\beta^n + \beta^{2n}) \\
&= \alpha^{2n+2} + 2(\alpha\beta)^{n+1} + \beta^{2n+2} - \alpha^{2n} - 2(\alpha\beta)^n - \beta^{2n} \\
&= \alpha^{2n+2} + 2(-1)^{n+1} + \beta^{2n+2} - \alpha^{2n} + 2(-1)^{n+1} - \beta^{2n} && \text{since } \alpha\beta = -1 \\
&= (\alpha^{2n+2} + \beta^{2n+2}) - (\alpha^{2n} + \beta^{2n}) + 4(-1)^{n+1} \\
&= L_{2n+2} - L_{2n} + 4(-1)^{n+1} \\
&= L_{2n+1} + 4(-1)^{n+1} && \text{by Equation (6)} \\
&= \alpha^{2n+1} + \beta^{2n+1} + 4(-1)^{n+1} \\
&= \alpha^{2n+1} + \beta^{2n+1} + 4(-1)^{n-1} \\
&= \alpha^{2n+1} + (\alpha^3 + \beta^3)(-1)^{n-1} + \beta^{2n+1} && \text{since } \alpha^3 + \beta^3 = 4 \\
&= \alpha^{2n+1} + (\alpha^3 + \beta^3)(\alpha\beta)^{n-1} + \beta^{2n+1} && \text{since } \alpha\beta = -1 \\
&= \alpha^{2n+1} + \alpha^2(\alpha\beta)^{n-1} + \beta^3(\alpha\beta)^{n-1} + \beta 2n + 1 \\
&= \alpha^{2n+1} + \beta^{n-1}\alpha^{n+2} + \alpha^{n-1}\beta^{n+2} + \beta^{2n+1} \\
&= (\alpha^{n-1} + \beta^{n-1})(\alpha^{n+2} + \beta^{n+2}) \\
&= L_{n-1}L_{n+2}
\end{aligned}$$

Hence $L_{n+1}^2 - L_n^2 = L_{n-1}L_{n+2}$ for all $n \geq 2$. □

Remark 8.7. Consider the following alternative proof of Proposition 8.6:

Proof. Let $n \geq 2$ be given.

WWTS: $L_{n+1}^2 - L_n^2 = L_{n-1}L_{n+2}$

Consider the following sequence of equalities:

$$\begin{aligned}
L_{n+1}^2 - L_n^2 &= (L_{n+1} - L_n)(L_{n+1} + L_n) \\
&= L_{n-1}L_{n+2} && \text{by Definition 7 and 5.}
\end{aligned}$$

Hence $L_{n+1}^2 - L_n^2 = L_{n-1}L_{n+2}$ for all $n \geq 2$. □

Proposition 8.8 (Exercise 18a from 14.3 [1]). *For all $n \geq 2$, the following identity holds:*

$$L_n = F_{n+1} + F_{n-1} = F_n + 2F_{n-1}.$$

Proof. We induct on n.

Base Cases: ($n = 2$) Observe that $F_2 + 2F_{2-1} = F_2 + 2F_1 = 1 + 2 = 3 = L_2$.
Observe that $F_3 + 2F_{3-1} = 2 + 2(1) = 4 = L_3$.
Thus, our base case is proven. ✓ □

Proposition 8.9 (Exercise 18b from 14.3 [1]). *For all $n \geq 3$, the following identity holds:*

$$L_n = F_{n+2} - F_{n-2}.$$

Proof. Let $n \geq 3$ be given.

WWTS: $L_n = F_{n+2} - F_{n-2}$.

Observe the following sequence of equalities:

$$\begin{aligned} L_n &= F_{n-1} + F_{n+1} && \text{by Proposition 8.8} \\ &= (F_n - F_{n-2}) + (F_{n+2} - F_n) && \text{by Equation (4)} \\ &= F_{n+2} - F_{n-2}. \end{aligned}$$

Hence $L_n = F_{n+2} - F_{n-2}$ holds for all $n \geq 3$. □

Proposition 8.10 (Exercise 18c from 14.3 [1]). *For all $n \geq 1$, the following identity holds::*

$$F_{2n} = F_n L_n.$$

Proof. Let $n \geq 1$ be given.

WWTS: $F_{2n} = F_n L_n$.

Consider the following sequence of equalities:

$$\begin{aligned} F_{2n} &= F_{n+n} \\ &= F_{n-1}F_n + F_nF_{n+1} && \text{by Proposition 4.10} \\ &= F_n(F_{n-1} + F_{n+1}) \\ &= F_nL_n && \text{by Proposition 8.8.} \end{aligned}$$

Hence $F_{2n} = F_nL_n$ for all $n \geq 1$. □

Proposition 8.11 (Exercise 18d from 14.3 [1])**.** *For all $n \geq 2$, the following identity holds:*

$$L_{n+1} + L_{n-1} = 5F_n.$$

Proof. Let $n \geq 2$ be given.

WWTS: $L_{n+1} + L_{n-1} = 5F_n$.

Consider the following sequence of equalities:

$$\begin{aligned}
L_{n+1} + L_{n-1} &= 5F_n \\
&= (F_{n+2} + F_n)(F_n + F_{n-2}) \\
&= 2F_n + F_{n+2} + F_{n-2} \\
&= 2F_n + (F_n + F_{n+1}) + (F_n - F_{n-1}) \qquad \text{by Equation (1) and Equation (4)} \\
&= 2F_n + 2F_n + F_{n+1} - F_{n-1} \\
&= 2F_n + 2F_n + F_n \qquad \text{by Equation (3)} \\
&= 5F_n.
\end{aligned}$$

Hence, we may conclude $L_{n+1} + L_{n-1} = 5F_n$ for all $n \geq 2$. □

Proposition 8.12 (Exercise 18e from 14.3 [1])**.** *For all $n \geq 2$, the following identity holds:*

$$L_n^2 = F_n^2 - 4F_{n+1}F_{n-1}.$$

Proof. Let $n \geq 2$ be given.

WWTS: $L_n^2 = F_n^2 - 4F_{n+1}F_{n-1}$.

Observe the following sequence of equalities:

$$\begin{aligned}
F_n^2 - 4F_{n+1}F_{n-1} &= F_n^2 + 4(F_n^2 + (-1)^n) \qquad \text{by Proposition 4.16} \\
&= 5F_n^2 + 4(-1)^n \\
&= 5\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)^2 + 4(-1)^n \\
&= \cancel{5} \cdot \frac{\alpha^{2n} + \beta^{2n} - 2(\alpha\beta)^n}{\cancel{(\alpha-\beta)^2}} + 4(-1)^n \qquad \text{since } (\alpha-\beta)^2 = 5 \\
&= \alpha^{2n} + \beta^{2n} - 2(\alpha\beta)^n + 4(\alpha\beta)^n \qquad \text{since } \alpha\beta = -1 \\
&= \alpha^{2n} + 2(\alpha\beta)^n + \beta^{2n}
\end{aligned}$$

$$= (\alpha^n + \beta^n)^2$$
$$= L_n^2.$$

Hence $L_n^2 = F_n^2 - 4F_{n+1}F_{n-1}$ holds for all $n \geq 2$. $\square$

Proposition 8.13 (Exercise 18f from 14.3 [1]). *For all $n, m \geq 1$, the following identity holds:*

$$2F_{m+n} = F_m L_n + L_m F_n.$$

Proof. In order to complete this proof, we fix m and induct on n.

Base Cases: $(n = 1)$ and $(n = 2)$

$(n = 1)$ Observe the following sequence of equalities:

$$\begin{aligned} F_m L_1 + L_m F_1 &= F_m + L_m \\ &= F_m + L_m \\ &= F_m + F_{m-1} + F_{m+1} && \text{from Proposition 8.8} \\ &= 2F_{m+1} \end{aligned}$$

$(n = 2)$ Observe the following sequence of equalities:

$$\begin{aligned} F_m L_2 + L_m F_2 &= 3F_m + L_m \\ &= 3F_m + F_{m-1} + F_{m+1} && \text{from Proposition 8.8} \\ &= 2F_m + (F_m + F_{m-1}) + F_{m+1} \\ &= 2F_m + 2F_{m+1} && \text{from Equation (1)} \\ &= 2F_{m+2} && \text{from Equation (1).} \end{aligned}$$

Therefore we have proved the two Base Cases of when $n = 1, 2$.

Induction Hypotheses: Suppose $2F_{m+k-1} = F_m L_{k-1} + L_m F_{k-1}$ and $2F_{m+k} = F_m L_k + L_m F_k$ for some $k \geq 2$.

WWTS: $2F_{m+k+1} = F_m L_{k+1} + L_m F_{k+1}.$

Consider the following sequence of equalities:

$$\begin{aligned} 2F_{m+k+1} &= 2F_{m+k} + 2F_{m+k-1} && \text{by Equation (1)} \\ &= F_m L_{k-1} + L_m F_{k-1} + F_m L_k + L_m F_k && \text{by Induction Hypotheses} \\ &= L_m F_{k-1} + L_m F_k + F_m L_{k-1} + F_m L_k \\ &= L_m(F_{k-1} + F_k) + F_m(L_{k-1} + L_k) \\ &= L_m F_{k+1} + F_m L_{k+1} && \text{by Equation (1) and Definition 5.} \end{aligned}$$

Therefore, we have proven that $2F_{m+n} = F_m L_n + L_m F_n$ for all $n, m \geq 1$. $\square$

Proposition 8.14 (Exercise 18g from 14.3 [1]). *For all $n \geq 1$, the following identity holds:*

$$\gcd(F_n, L_n) = 1 \textit{ or } 2.$$

Proof. Let $n \geq 1$ be given.

WWTS: $\gcd(F_n, L_n) = 1$ or 2.

According to Proposition 8.8, we can rewrite $\gcd(F_n, L_n)$ as $\gcd(F_n, F_n + 2F_{n-1})$. Then by Lemma 6.1 we can rewrite $\gcd(F_n, F_n + 2F_{n-1})$ as $\gcd(F_n, 2F_{n-1})$. Moreover, by Lemma 6.2, we know $\gcd(F_n, 2F_{n-1}) = \gcd(F_n, 2)$ since $\gcd(F_n, F_{n+1}) = 1$ by Corollary 7.6. Hence the inequality $\gcd(F_n, L_n) = \gcd(F_n, 2) \leq 2$ is forced. Observe that if F_n is even (respectively, odd), then $\gcd(F_n, L_n) = 2$ (respectively, $\gcd(F_n, L_n) = 1$). Hence $\gcd(F_n, L_n) = 1$ or 2 for all $n \geq 1$. □

Proposition 8.15 (Exercise 20a from 14.3 [1]). *For all $n \geq 1$, the following identity holds:*

$$L_n^2 = L_{2n} + 2(-1)^n.$$

Proof. Let $n \geq 1$ be given.

WWTS: $L_n^2 = L_{2n} + 2(-1)^n$.

Observe the following sequence of equalities:

$$\begin{aligned}
L_n^2 &= (\alpha^n + \beta^n)^2 \\
&= \alpha^{2n} + 2(\alpha\beta)^n + \beta^{2n} \\
&= \left(\alpha^{2n} + \beta^{2n}\right) + 2(\alpha\beta)^n \\
&= L_{2n} + 2(-1)^n && \text{since } \alpha\beta = -1.
\end{aligned}$$

Hence $L_n^2 = L_{2n} + 2(-1)^n$ holds for all $n \geq 1$. □

Proposition 8.16 (Exercise 20b from 14.3 [1]). *For all $n \geq 1$, the following identity holds:*

$$L_n L_{n+1} - L_{2n+1} = (-1)^n.$$

Proof. Let $n \geq 1$ be given.

WWTS: $L_n L_{n+1} - L_{2n+1} = (-1)^n$.

Consider the following sequence of equalities:

$$\begin{aligned}
L_nL_{n+1} - L_{2n+1} &= (\alpha^n + \beta^n)(\alpha^{n+1} + \beta^{n+1}) - (\alpha^{2n+1} + \beta^{2n+1}) \\
&= \alpha^{2n+1} + \alpha^{n+1}\beta^n + \beta^{n+1}\alpha^n + \beta^{2n+1} - \alpha^{2n+1} - \beta^{2n+1} \\
&= \cancel{\alpha^{2n+1}} + \alpha^{n+1}\beta^n + \beta^{n+1}\alpha^n + \cancel{\beta^{2n+1}} \cancel{-\alpha^{2n+1}} - \cancel{\beta^{2n+1}} \\
&= \alpha^{n+1}\beta^n + \beta^{n+1}\alpha^n \\
&= \alpha(\alpha\beta)^n + \beta(\alpha\beta)^n \\
&= (\alpha\beta)^n(\alpha + \beta) \\
&= (-1)^n \cdot 1 \\
&= (-1)^n
\end{aligned}$$

Hence we have proven that $L_nL_{n+1} - L_{2n+1} = (-1)^n$ for all $n \geq 1$. □

Proposition 8.17 (Exercise 20c from 14.3 [1])**.** *For all $n \geq 2$, the following identity holds:*

$$L_n^2 - L_{n-1}L_{n+1} = 5(-1)^n.$$

Proof. Let $n \geq 2$ be given.

WWTS: $L_n^2 - L_{n-1}L_{n+1} = 5(-1)^n$.

$$\begin{aligned}
L_n^2 - L_{n-1}L_{n+1} &= (\alpha^n + \beta^n)^2 - (\alpha^{n-1} + \beta^{n-1})(\alpha^{n+1} + \beta^{n+1}) \\
&= (\alpha^n + \beta^n)^2 - \alpha^{2n} - \alpha^{n-1}\beta^{n+1} - \beta^{n-1}\alpha^{n+1} - \beta^{2n} \\
&= \alpha^{2n} + 2\alpha^n\beta^n + \beta^{2n} - \alpha^{2n} - \alpha^{n-1}\beta^{n+1} - \beta^{n-1}\alpha^{n+1} - \beta^{2n} \\
&= 2\alpha^n\beta^n - (\alpha^{n-1}\beta^{n+1})(\alpha^2 + \beta^2) \\
&= 2(-1)^n - (-1)^{n-1}L_2 \\
&= 2(-1)^n - 3(-1)^{n-1} \\
&= 2(-1)^n + 3(-1)(-1)^{n-1} \\
&= 2(-1)^n + 3(-1)^n \\
&= 5(-1)^n
\end{aligned}$$

Hence, we may conclude $L_n^2 - L_{n-1}L_{n+1} = 5(-1)^n$ for all $n \geq 2$.

□

Proposition 8.18 (Exercise 20d from 14.3 [1])**.** *For all $n \geq 3$, the following identity holds:*

$$L_{2n} + 7(-1)^n = L_{n-2}L_{n+2}.$$

Proof. Let $n \geq 3$ be given.

WWTS: $L_{2n} + 7(-1)^n = L_{n-2}L_{n+2}$.

Observe the following sequence of equalities:

$$\begin{aligned} L_{n-2}L_{n+2} &= (\alpha^{n-2} + \beta^{n-2})(\alpha^{n+2} + \beta^{n+2}) \\ &= \alpha^{2n} + \beta^{2n} + \alpha^{n+2}\beta^{n-2} + \alpha^{n-2}\beta^{n+2} \\ &= \alpha^{2n} + \beta^{2n} + \alpha^4 \cdot (\alpha\beta)^{n-2} + \beta^4 \cdot (\alpha\beta)^{n-2} \\ &= L_{2n} + (\alpha^4 + \beta^4) \cdot (\alpha\beta)^{n-2} \\ &= L_{2n} + L_4 \cdot (-1)^{n-2} && \text{since } \alpha\beta = -1 \\ &= L_{2n} + 7(-1)^{n-2}(-1)^2 && \text{since } L_4 = 7 \\ &= L_{2n} + 7(-1)^n. \end{aligned}$$

Hence $L_{2n} + 7(-1)^n = L_{n-2}L_{n+2}$ holds for all $n \geq 3$. □

Proposition 8.19 (Exercise 21a from 14.3 [1])**.** *For all $n \geq 1$, the following identity holds:*

$$L_n^2 - 5F_n^2 = 4(-1)^n.$$

Proof. Let $n \geq 1$ be given.

WWTS: $L_n^2 - 5F_n^2 = 4(-1)^n$.

Observe that it suffices to show that $L_n^2 - 4(-1)^n = 5F_n^2$. Consider the following sequence of equalities:

$$\begin{aligned} L_n^2 - 4(-1)^n &= (\alpha^n + \beta^n)^2 - 4(\alpha\beta)^n \\ &= \alpha^{2n} + 2(\alpha\beta)^n + \beta^{2n} - 4(\alpha\beta)^n \\ &= \alpha^{2n} - 2(\alpha\beta)^n + \beta^{2n} \\ &= (\alpha^n - \beta^n)^2 \\ &= \frac{5}{5}(\alpha^n - \beta^n)^2 \\ &= 5\frac{(\alpha^n - \beta^n)^2}{(\alpha - \beta)^2} && \text{since } (\alpha - \beta)^2 = 5 \\ &= 5F_n^2. \end{aligned}$$

Hence $L_n^2 - 5F_n^2 = 4(-1)^n$ for all $n \geq 1$. □

Proposition 8.20 (Exercise 21b from 14.3 [1])**.** *For all $n \geq 1$, the following identity holds:*

$$L_{2n+1} = 5F_nF_{n+1} + (-1)^n.$$

Proof. Let $n \geq 1$ be given.

WWTS: $L_{2n+1} = 5F_nF_{n+1} + (-1)^n$.

$$\begin{aligned}
L_{2n+1} - 5F_nF_{n+1} &= (\alpha^{2n+1} + \beta^{2n+1}) - 5\frac{(\alpha^n - \beta^n)(\alpha^{n+1} - \beta^{n+1})}{(\alpha - \beta)^2} \\
&= (\alpha^{2n+1} + \beta^{2n+1}) - (\alpha^n + \beta^n)(\alpha^{n+1} + \beta^{n+1}) \\
&= (\alpha^{2n+1} + \beta^{2n+1}) - (\alpha^n\alpha^{n+1} + \alpha^n\beta^{n+1} + \beta^n\alpha^{n+1} - \beta^n\beta^{n+1}) \\
&= (\alpha^{2n+1} + \beta^{2n+1}) - \alpha^{2n+1} + (\alpha^n\beta^n)(\alpha + \beta) - \beta^{2n+1} \\
&= (\alpha^n\beta^n)(\alpha + \beta) \qquad = (-1)^n(1) = (-1)^n
\end{aligned}$$

Hence, we may conclude $L_{2n+1} = 5F_nF_{n+1} + (-1)^n$ for all $n \geq 1$. ☐

Proposition 8.21 (Exercise 21c from 14.3 [1])**.** *For all $n \geq 2$, the following identity holds:*

$$L_n^2 - F_n^2 = 4F_{n+1}F_{n-1}.$$

Proof. This proposition is almost VERBATIM the statement of Proposition 8.12, so we refer the reader to that proposition for the proof. ☐

Proposition 8.22 (Exercise 21d from 14.3 [1])**.** *For all $m, n \geq 1$, the following identity holds:*

$$L_mL_n + 5F_mF_n = 2L_{m+n}.$$

Proof. Let $m, n \geq 1$ be given.

WWTS: $L_mL_n + 5F_mF_n = 2L_{m+n}$.

Consider the following sequence of equalities:

$$\begin{aligned}
&L_mL_n + 5F_mF_n \\
&\qquad = (\alpha^m + \beta^m)(\alpha^n + \beta^n) + 5(\frac{\alpha^m - \beta^m}{\alpha - \beta})(\frac{\alpha^n - \beta^n}{\alpha - \beta}) \\
&\qquad = \alpha^{m+n} + \alpha^m\beta^n + \beta^m\alpha^n + \beta^{m+n} + 5 \cdot \frac{\alpha^{m+n} - \beta^m\alpha^n - \alpha^m\beta^n + \beta^{m+n}}{(\alpha - \beta)^2}
\end{aligned}$$

$$
\begin{aligned}
&= \alpha^{m+n} + \alpha^m\beta^n + \beta^m\alpha^n + \beta^{m+n} + \cancel{5}\cdot\frac{\alpha^{m+n} - \beta^m\alpha^n - \alpha^m\beta^n + \beta^{m+n}}{\cancel{(\alpha-\beta)^2}} \\
&= \alpha^{m+n} + \alpha^m\beta^n + \beta^m\alpha^n + \beta^{m+n} + \alpha^{m+n} - \beta^m\alpha^n - \alpha^m\beta^n + \beta^{m+n} && \text{since } (\alpha-\beta)^2 = 5 \\
&= \alpha^{m+n} + \cancel{\alpha^m\beta^n} + \cancel{\beta^m\alpha^n} + \beta^{m+n} + \alpha^{m+n} \cancel{- \beta^m\alpha^n - \alpha^m\beta^n} + \beta^{m+n} \\
&= \alpha^{m+n} + \beta^{m+n} + \alpha^{m+n} + \beta^{m+n} \\
&= 2(\alpha^{m+n} + \beta^{m+n}) \\
&= 2L_{m+n}.
\end{aligned}
$$

Therefore, we have proven that $L_mL_n + 5F_mF_n = 2L_{m+n}$ for all $m, n \geq 1$. $\square$

Proposition 8.23 (Exercise 22 from 14.3 [1]). *For all $n \geq 2$, the following identity holds:*

$$L_{2^n} \equiv 7 \pmod{10}.$$

Proof. Let $n \geq 2$ be given. We induct on n.

Base Case: ($n = 2$) Our statement is as follows: $L_{2^2} = L_4 = 7 \equiv 7 \pmod{10}$. ✓

Induction Hypothesis: Suppose $L_{2^k} \equiv 7 \pmod{10}$ for some $k \geq 2$.

WWTS: $L_{2^{k+1}} \equiv 7 \pmod{10}$.

Consider the following sequence of equalities:

$$
\begin{aligned}
L_{2^{k+1}} &= L_{2^k \cdot 2} \\
&= L_{2^k}^2 - 2(-1)^{2^k} && \text{from Proposition 8.15} \\
&= L_{2^k}^2 - 2 \\
&\equiv 7^2 - 2 \pmod{10} && \text{by Induction Hypothesis} \\
&\equiv 47 \pmod{10} \\
&\equiv 7 \pmod{10}.
\end{aligned}
$$

Hence $L_{2^n} \equiv 7 \pmod{10}$ for all $n \geq 2$. $\square$

9 Our Own Results and Conjectures

aBa: [Also we should call these theorems instead of propositions since they are original results and not in Burton, Vorobiev, or elsewhere most likely.]

Theorem 9.1. *Let $n, r \in \mathbb{Z}$ such that $n \geq 0$ and $r \geq 1$. Then $F_{n+r} \equiv F_{r-1}F_n \pmod{F_r}$.*

Proof. Let $n, r \in \mathbb{Z}$ such that $n \geq 0$ and $r \geq 1$.

WWTS: $F_{n+r} \equiv F_{r-1}F_n \pmod{F_r}$.

It suffices to show that the difference $F_{n+3} \equiv F_n$ is divisible by 2. To this end, observe the following:

$$\begin{aligned} F_{n+r} &= F_{r+n} \\ &= F_{r-1}F_n + F_rF_{n+1} && \text{by Proposition 4.10} \end{aligned}$$

Thus we have $F_{n+r} - F_{r-1}F_n = F_rF_{n+1}$. Hence $F_{n+r} \equiv F_{r-1}F_n \pmod{F_r}$ as desired. □

Theorem 9.2. *Let $n \geq m \geq 1$. Then $F_{2n-m} = F_{n-m-1}F_n + F_{n-m}F_{n+1}$.*

Proof. The following holds by Proposition 4.10.

$$F_{2n-m} = F_{(n-m)+n} = F_{n-m-1}F_n + F_{n-m}F_{n+1}.$$

Hence $F_{2n-m} = F_{n-m-1}F_n + F_{n-m}F_{n+1}$ for all $n \geq m \geq 1$. □

Corollary 9.3. *I (aBa) am writing this loosely for now. But maybe we can prove a corollary to the above theorem to prove that if $3 \mid m + n$ then the value F_{2n-m} is even.*

Theorem 9.4. *The numbers 17 and 34 divide F_{9n} for all $n \geq 1$*

Proof. Let $n \geq 1$ be given.

WWTS: $17 \mid F_{9n}$ and $34 \mid F_{9n}$.

Using Corollary 7.5, it can be seen that since $9 \mid 9n$, then $F_9 \mid F_{9n}$. Because $F_9 = 34$, it follows that 34 divides all Fibonacci numbers of the form F_{9n}. Since $34 = 2 \cdot 17$, it also follows that 17 divides F_{9n} for all $n \geq 1$. □

Theorem 9.5. *The number 17 doesn't divide F_{3n+2} for all $n \geq 2$.*

Proof. Let $n \geq 2$ be given. Suppose by way of contradiction that 17 divides F_{3n+2}. Then, 17 divides F_{9n+2} since $9\mathbb{Z}+2 \subset 3\mathbb{Z}+2$. Consider the following use of Equation (1):

$$F_{9n+2} = F_{9n+1} + F_{9n}.$$

Notice 17 divides F_{9n+2} by assumption, and 17 divides F_{9n} by Theorem 9.4. Thus, 17 must divide F_{9n+1}. Again, consider the following use of Equation (1):

$$F_{9n+1} = F_{9n} + F_{9n-1}.$$

Since 17 divides F_{9n+1} from above and 17 divides F_{9n} from Theorem 9.4, then 17 must divide F_{9n-1}. Again, consider the following use of Equation (1):

$$F_{9n} = F_{9n-1} + F_{9n-2}.$$

Since 17 divides F_{9n} from Theorem 9.4 and 17 divides F_{9n-1} from above, then 17 must divide F_{9n-2}. Again, consider the following use of Equation (1):

$$F_{9n-1} = F_{9n-2} + F_{9n-3}.$$

Since 17 divides F_{9n-1}, and 17 divides F_{9n-2}, then 17 must divide F_{9n-3}. Again, consider the following use of Equation (1):

$$F_{9n-2} = F_{9n-3} + F_{9n-4}.$$

Since 17 divides F_{9n-2} and 17 divides F_{9n-3}, then 17 must divide F_{9n-4}. Again, consider the following use of Equation (1):

$$F_{9n-3} = F_{9n-4} + F_{9n-5}.$$

Since 17 divides F_{9n-3} and 17 divides F_{9n-4}, then 17 must divide F_{9n-5}. Again, consider the following use of Equation (1):

$$F_{9n-4} = F_{9n-5} + F_{9n-6}.$$

Since 17 divides F_{9n-4} and 17 divides F_{9n-5}, then 17 must divide F_{9n-6}. Therefore, 17 divides all terms from the following list: $F_{9n-6}, F_{9n-5}, F_{9n-4}, F_{9n-3}, F_{9n-2}, F_{9n-1}, F_{9n}, F_{9n+1}, F_{9n+2}$. Hence 17 divides F_n for all $n \geq 12$. However, this is a contradiction since 17 doesn't divide all Fibonacci numbers greater than or equal to F_{12}. For example, notice $F_{12} = 144 = 2^4 \cdot 3^2$, which is not divisible by 17. Therefore, 17 doesn't divide F_{3n+2}. □

Theorem 9.6. *The following identity holds:* $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \alpha$.

Proof. Consider the following limits:

$$\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \lim_{n\to\infty} \frac{F_{n+1}}{F_n} = L.$$

Hence, by Equation 1 it follows that

$$L = \lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \lim_{n\to\infty} \frac{F_n + F_{n-1}}{F_n} = \lim_{n\to\infty} \frac{F_n}{F_n} + \lim_{n\to\infty} \frac{F_{n-1}}{F_n} = 1 + \frac{1}{L}.$$

Therefore, we must solve the equation $L = 1 + \frac{1}{L}$. Multiplying both sides of this equation by L, it follows that $L^2 = L + 1$. Therefore, it suffices to solve the quadratic $L^2 - L - 1 = 0$. From Proposition 2, we have previously solved for this solution. Hence $L = \alpha, \beta$. However, β is an extraneous solution, hence $L = \alpha$. Therefore, it is proven that $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \alpha$. ☐

Theorem 9.7. *For all $m \geq 0$, the following identitiy holds:*

$$\lim_{n\to\infty} \frac{\prod_{i=1}^{m+1} F_{n+i}}{\prod_{i=0}^{m} F_{n+i}} = \alpha^{m+1}.$$

Proof. Consider the following simplification of our LHS:

$$\begin{aligned}
\lim_{n\to\infty} \frac{\prod_{i=1}^{m+1} F_{n+i}}{\prod_{i=0}^{m} F_{n+i}} &= \lim_{n\to\infty} \frac{F_{n+1}F_{n+2}F_{n+3}\cdots F_{n+m+1}}{F_nF_{n+1}F_{n+2}\cdots F_{n+m}} \\
&= \lim_{n\to\infty} \frac{\cancel{F_{n+1}F_{n+2}F_{n+3}\cdots}F_{n+m+1}}{F_n\cancel{F_{n+1}F_{n+2}\cdots F_{n+m}}} \\
&= \lim_{n\to\infty} \frac{F_{n+m+1}}{F_n}.
\end{aligned}$$

Therefore, it suffices to establish $\lim_{n\to\infty} \frac{F_{n+m+1}}{F_n} = \alpha^{m+1}$ for all $m \geq 0$. In order to prove our statement, we induct on m.

Base Cases: $(n = 0)$ and $(n = 1)$
$(n = 0)$ See Theorem 9.6. $(n = 1)$ Observe the following sequence of equalities:

$$\begin{aligned}
\lim_{n\to\infty} \frac{F_{n+2}}{F_n} &= \lim_{n\to\infty} \frac{F_{n+1} + F_n}{F_n} \\
&= \lim_{n\to\infty} \frac{F_{n+1}}{F_n} + \lim_{n\to\infty} \frac{F_n}{F_n} \\
&= \alpha + 1 && \text{by Theorem 9.6} \\
&= \alpha F_2 + F_1 \\
&= \alpha^2 && \text{by Proposition 5.1.}
\end{aligned}$$

Therefore we have proved the two Base Cases of when $m = 0, 1$.

Induction Hypotheses: Suppose $\lim_{n\to\infty} \frac{F_{n+k+1}}{F_n} = \alpha^{k+1}$ and $\lim_{n\to\infty} \frac{F_{n+k+1+1}}{F_n} = \alpha^{k+1+1}$ for some $k \geq 0$.

WWTS: $\lim_{n\to\infty} \frac{F_{n+k+2+1}}{F_n} = \alpha^{k+2+1}$.

Consider the following sequence of equalities:

$$\begin{aligned}
\lim_{n\to\infty}\frac{F_{n+k+2+1}}{F_n} &= \lim_{n\to\infty}\frac{F_{n+k+2}+F_{n+k+1}}{F_n} && \text{by Equation (1)}\\
&= \lim_{n\to\infty}\frac{F_{n+k+2}}{F_n}+\lim_{n\to\infty}\frac{F_{n+k+1}}{F_n}\\
&= \alpha^{k+1}+\alpha^{k+2} && \text{by Induction Hypotheses}\\
&= (\alpha\cdot F_{k+1}+F_k)+(\alpha\cdot F_{k+2}+F_{k+1}) && \text{by Propostion 5.1}\\
&= \alpha(F_{k+1}+F_{k+2})+(F_k+F_{k+1})\\
&= \alpha(F_{k+3})+F_{k+2} && \text{by Equation (1).}\\
&= \alpha^{k+3} && \text{by Proposition 5.1.}
\end{aligned}$$

Therefore, we have proven that $\lim_{n\to\infty}\frac{\prod_{i=1}^{m+1}F_{n+i}}{\prod_{i=0}^{m}F_{n+i}}=\alpha^{m+1}$ for all $m\geq 0$. □

Proposition 9.8. *The following identity holds for all $n \geq 0$:*

$$F_{5n+3}\equiv L_n \pmod{10}.$$

Proof. Let $n\geq 0$ be given. We induct on n.

Base Cases: ($n=0$) Our statement is as follows: $F_{5(0)+3}=F_3=2\equiv L_0 \pmod{10}$.
($n=1$) The left side of our statement is as follows: $F_{5(1)+3}=F_8=21\equiv 1 \pmod{10}$. The right side of our statement is as follows: $L_1=1\equiv 1 \pmod{10}$.

Induction Hypotheses: Suppose the following for statements for some $k\geq 0$:

$$\begin{aligned}
F_{5k+3}&\equiv L_k \pmod{10}\\
F_{5k+8}&\equiv L_{k+1} \pmod{10}.
\end{aligned}$$

WWTS: $F_{5k+13}\equiv L_{k+2} \pmod{10}$.

Consider the following sequence of congruences:

$$\begin{aligned}
F_{5k+13} &= F_{5k-1}F_{13}+F_{5k}F_{14} && \text{by Proposition 4.10}\\
&= 233F_{5k-1}+377F_{5k}\\
&\equiv 3F_{5k-1}+7F_{5k} \pmod{10}\\
&\equiv 23F_{5k-1}+37F_{5k} \pmod{10}\\
&\equiv (21+2)F_{5k-1}+(3+34)F_{5k} \pmod{10}\\
&\equiv 21F_{5k-1}+2F_{5k-1}+3F_{5k}+34F_{5k} \pmod{10}\\
&\equiv F_8F_{5k-1}+F_3F_{5k-1}+F_4F_{5k}+F_9F_{5k} \pmod{10}
\end{aligned}$$

$$
\begin{aligned}
&\equiv F_8F_{5k-1} + F_9F_{5k} + F_3F_{5k-1} + F_4F_{5k} \pmod{10} \\
&\equiv F_{5k+8} + F_{5k+3} \pmod{10} && \text{by Proposition 4.10} \\
&\equiv F_{5k+8} \pmod{10} + F_{5k+3} \pmod{10} \\
&\equiv L_{k+1} \pmod{10} + L_k \pmod{10} && \text{by Induction Hypotheses} \\
&\equiv L_{k+1} + L_k \pmod{10} \\
&\equiv L_{k+2} \pmod{10} && \text{by Equation (5).}
\end{aligned}
$$

Therefore $F_{5n+3} \equiv L_n (\text{mod } 10)$ for all $n \geq 0$. □

Proposition 9.9. *The following identity holds for all $r \geq 1$:*

$$\frac{\operatorname{lcm}(r, 60)}{r} = n$$

where n is the size of the tuple generated by a jump of size r.

Proof. Let r begin at some starting position k. Observe that only a jump of size $60J$ will result in traveling back to k and beginning the tuple over again, for some $J \in \mathbb{Z}$. Observe that $\frac{\operatorname{lcm}(r,60)}{r}$ describes how many terms must be jumped in order to reach a multiple of 60 for the first time divided by the size of the jump. Thus, resulting in the number of iterations of a jump of size r required to land back at k for the first time, which is equivalent to the size of the tuple generated by r. □

Theorem 9.10.

$$\mathcal{F}_{m,a+b} \equiv \mathcal{F}_{m,a-1}\mathcal{F}_{m,b} + \mathcal{F}_{m,a}\mathcal{F}_{m,1+b} \pmod{m}.$$

Proof. Let a be fixed. We induct on b.

Base Case: ($b = 0$) Observe that the left hand side equals $\mathcal{F}_{m,a}$. Observe that the right hand side is equal to the following:

$$\mathcal{F}_{m,a-1}\mathcal{F}_{m,0} + \mathcal{F}_{m,a}\mathcal{F}_{m,1} = 0 + \mathcal{F}_{m,a} \cdot 1 = \mathcal{F}_{m,a}.$$

Thus our base case is proven. ✓

Base Case: ($b = 1$) Observe that the left hand side equals $\mathcal{F}_{m,a+1}$. Observe that the right hand side is equal to the following:

$$\mathcal{F}_{m,a-1}\mathcal{F}_{m,1} + \mathcal{F}_{m,a}\mathcal{F}_{m,2} = \mathcal{F}_{m,a-1} + \mathcal{F}_{m,a} \cdot 1 = \mathcal{F}_{m,a+1}.$$

Thus our base case is proven. ✓

Induction Hypotheses: Assume

$$\mathcal{F}_{m,a+k} \equiv \mathcal{F}_{m,a-1}\mathcal{F}_{m,k} + \mathcal{F}_{m,a}\mathcal{F}_{m,k+1} \pmod{m}$$

and

$$\mathcal{F}_{m,a+k-1} \equiv \mathcal{F}_{m,a-1}\mathcal{F}_{m,k-1} + \mathcal{F}_{m,a}\mathcal{F}_{m,k} \pmod{m}$$

for some $k \geq 1$.

WWTS: $\mathcal{F}_{m,a+k+1} \equiv \mathcal{F}_{m,a-1}\,\mathcal{F}_{m,k+1} + \mathcal{F}_{m,a}\,\mathcal{F}_{m,k+2} \pmod{m}$.

Observe the following sequence of equalities:

$$
\begin{aligned}
&\mathcal{F}_{m,a-1}\,\mathcal{F}_{m,k+1} + \mathcal{F}_{m,a}\,\mathcal{F}_{m,k+2} \\
&\qquad \equiv \mathcal{F}_{m,a-1}(\mathcal{F}_{m,k} + \mathcal{F}_{m,k-1}) + \mathcal{F}_{m,a}(\mathcal{F}_{m,k+1} + \mathcal{F}_{m,k}) \pmod{m} \\
&\qquad \equiv \mathcal{F}_{m,a-1}\,\mathcal{F}_{m,k} + \mathcal{F}_{m,a-1}\,\mathcal{F}_{m,k-1} + \mathcal{F}_{m,a}\,\mathcal{F}_{m,k+1} + \mathcal{F}_{m,a}\,\mathcal{F}_{m,k} \pmod{m} \\
&\qquad \equiv (\mathcal{F}_{m,a-1}\,\mathcal{F}_{m,k} + \mathcal{F}_{m,a}\,\mathcal{F}_{m,k+1}) + (\mathcal{F}_{m,a-1}\,\mathcal{F}_{m,k-1} + \mathcal{F}_{m,a}\,\mathcal{F}_{m,k}) \pmod{m} \\
&\qquad \equiv \mathcal{F}_{m,a+k} + \mathcal{F}_{m,a+k-1} \pmod{m} \qquad \text{By Induction Hypothesis} \\
&\qquad \equiv \mathcal{F}_{m,a+k+1} \pmod{m}.
\end{aligned}
$$

Hence it is proven that $\mathcal{F}_{m,a+b} \equiv \mathcal{F}_{m,a-1}\,\mathcal{F}_{m,b} + \mathcal{F}_{m,a}\,\mathcal{F}_{m,b+1} \pmod{m}$. □

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